Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: FY BSc

Subject : Numerical methods

Subject Code: PUSAS201

Chapter: Unit 2 Chapter 2

Chapter Name: Determinants & vectors

Today's Agenda

1. Determinant
    1. Determinant of a 2x2 matrix
    2. Determinant of a 3x3 matrix
    3. Determinant of any square matrix
    4. Minors and co-factors of a matrix
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2. Vectors
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    4. Addition and Scalar multiplication of vectors
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1 Determinant

- The determinant of a matrix is a special number that can be calculated from a square matrix.
- The determinant tells us things about the matrix that are useful in systems of linear equations, helps us find the inverse of a matrix, is useful in calculus and more.
- The symbol for determinant is two vertical lines either side. |A| means the determinant of the matrix A
- The matrix must be square (i.e. have the same number of rows as columns). Then it is just basic arithmetic.

1. For a 2 × 2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is: $|A| = ad - bc$. "The determinant of A equals a times d minus b times c"

Just remember: Multiplying downwards is positive, multiplying upwards is negative.

1.1 Determinant of a 2x2 matrix

If:

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The determinant is: $|A| = ad - bc$. "The determinant of A equals a times d minus b times c"

Just remember: Multiplying downwards is positive, multiplying upwards is negative.

1.2 Determinant of a 3x3 matrix

The determinant of the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

is $|A| = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} - a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$

The computation of the 3 × 3 determinant takes the form

$$|A| = a_{11} \begin{vmatrix} * & * & * \\ * & a_{22} & a_{23} \\ * & a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} * & * & * \\ a_{21} & * & a_{23} \\ a_{31} & * & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} * & * & * \\ a_{21} & a_{22} & * \\ a_{31} & a_{32} & * \end{vmatrix}$$

Where the first 2 × 2 determinant is obtained by deleting the first row and first column, the second by deleting the first row and second column and the third by deleting the first row and third column

1.2 Example

Find the determinant of the matrix

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$$

The determinant is given by $|A| = 2\begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} - 1\begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix}$

$$= (2)[3 - (-12)] - (1)(9 - 20) - 1(-9 - 5)$$

$$= 30 + 11 + 14$$

$$= 55$$

1.3 Determinant of any Square Matrix

If A is an $n \times n$ matrix, then

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n} = \sum_{k=1}^{n} a_{1k}C_{1k}$$

Thus, the determinant of any square matrix of any size can be found by expanding along any row or column.

1.4 Minors and Co-factors of a Matrix

- If A is a square matrix, then the minor M_{ij}, associated with the entry a_{ij}, is the determinant of the $(n - 1) \times (n - 1)$ matrix obtained by deleting row i and column j from the matrix A.

- The cofactor of $_{ij}$ is $C_{ij} = (-1)^{i+j} M_{ij}$

- The determinant of A is given by the cofactor expansion

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

1.4 Example

Find the minors of the following matrix and the determinant using the cofactor expansion

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 1 & 4 \\ 5 & -3 & 3 \end{bmatrix}$$

The minors are as follows:

$$M_{11} = \begin{vmatrix} 1 & 4 \\ -3 & 3 \end{vmatrix} \; M_{12} = \begin{vmatrix} 3 & 4 \\ 5 & 3 \end{vmatrix} \; M_{11} = \begin{vmatrix} 3 & 1 \\ 5 & -3 \end{vmatrix}$$

The determinant of A $= 2\,(-1)^2\,(15) + 1\,(-1)^3\,(-11) - 1\,(-1)^4\,(-14)$

$= 30 + 11 + 15$

$= 55$

1.5 Properties of Determinants

Let A be a square matrix

1. If two rows of A are interchanged to produce a matrix B, then det (B) = −det (A)
2. If a multiple of one row of A is added to another row to produce a matrix B, then det (A) = det(B)
3. If a row of A is multiplied by a real number α to produce a matrix B, then det (A) = det (B)

Let A and B be $n \times n$ matrices and α a real number.

1. det (AB) = det (A) det (B)
2. det (αA) = α^n det (A)
3. det (A^t) = det (A)
4. If A has a row (or column) of all zeros, then det(A) = 0
5. If A has two equal rows (or columns), then det(A)=0
6. If A has a row(or column) that is a multiple of another row or column, then det(A)=0.

1.6 Theorem

Theorem

A square matrix A is invertible if and only if $\det(A) \neq 0$

Corollary

Let A be an invertible matrix. Then

1. $\det(A^{-1}) = 1/\det A$
2. The linear system $A\mathbf{x} = \mathbf{b}$ has a unique solution for every vector b
3. The determinant of the matrix A is nonzero.

A matrix that is not invertible is called a singular matrix.

Thus, if |A|=0, the matrix A is a singular matrix.

1.7 Cramer's Rule

Let A be an $n \times n$ invertible matrix, and let b be a column vector with n components.
Let A_i be the matrix obtained by replacing the ith column of A with $\mathbf{b}$.

If $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}$ is the unique solution to the linear system $A\mathbf{x} = \mathbf{b}$, then

$$\boldsymbol{x_i} = \frac{det(A_i)}{det(A)} \quad for \quad i = 1,2, \dots \dots n$$

1.7 Example

Solve the linear system

$$
\begin{aligned}
2x + 3y - z &= 2 \\
3x - 2y + z &= -1 \\
-5x - 4y + 2z &= 3
\end{aligned}
$$

The determinant of the coefficient matrix is given by

$$\begin{vmatrix} 2 & 3 & -1 \\ 3 & -2 & 1 \\ -5 & -4 & 2 \end{vmatrix} = -11$$

By Cramer's rule, the solution to the system is

$$x = -\frac{1}{11}\begin{vmatrix} 2 & 3 & -1 \\ -1 & -2 & 1 \\ 3 & -4 & 2 \end{vmatrix} = -\frac{5}{11} \quad y = -\frac{1}{11}\begin{vmatrix} 2 & 2 & -1 \\ 3 & -1 & 1 \\ -5 & 3 & 2 \end{vmatrix} = \frac{36}{11}\,\frac{5}{11}$$

$$z = -\frac{1}{11}\begin{vmatrix} 2 & 3 & 2 \\ 3 & -2 & -1 \\ -5 & -4 & 3 \end{vmatrix} = \frac{76}{11}$$

2 Vectors

- A vector is an $n \times 1$ matrix.
- The entries of a vector are called its components.

For a given matrix A, it is convenient to refer to its row vectors and its column vectors

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 0 & 1 \\ 4 & -1 & 2 \end{bmatrix}$$

Then the column vectors of A are:

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix}, \text{and} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

While the row vectors of A, written vertically, are:

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, and \begin{bmatrix} 4 \\ -1 \\ 2 \end{bmatrix}$$

2.1 Dot Product of Two Vectors

Given two vectors

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ . \\ u_n \end{bmatrix} \; and \; \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ . \\ . \\ . \\ v_n \end{bmatrix}$$

The dot product is defined by

$$\mathbf{u}.\mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i$$

2.2 Vectors in $\mathbb{R}^n$

Euclidean 2-space, denoted by $\mathbb{R}^2$,is the set of all vectors with two entries, that is,

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| x_1, x_2 \text{ are real numbers} \right\}$$

Similarly, Euclidean 3-space, denoted by $\mathbb{R}^3$,is the set of all vectors with three entries, that is,

$$\mathbb{R}^3 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle| x_1, x_2, x_3 \text{ are real numbers} \right\}$$

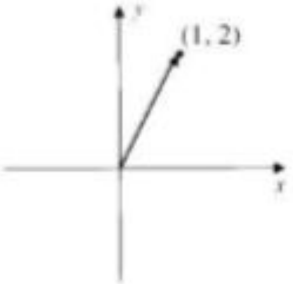
Euclidean n-space, denoted by $\mathbb{R}^n$, or simply n-space, is defined by

$$\mathbb{R}^2 = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \middle| x_i \in \mathbb{R} \text{ for } i = 1,2, \dots n \right\}$$

The entries of a vector are called the **components** of the vector

2.3 Geometric Interpretation

- Geometrically in $\mathbb{R}^2$ and $\mathbb{R}^3$, a vector is a directed line segment from the origin to the point whose origins are equal to the components of the vector.
- The vector $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ in $\mathbb{R}^2$ is the directed line segment from the origin (0,0) to the point (1,2)
- Point (0,0) is the **initial point**, point (1,2) is the **terminal point**
- The length of the vector is length of the line segment from the initial point to the terminal point
- The length of $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is $\sqrt{1^2 + 2^2} = 5$
- Since vectors are matrices, two vectors are equal if their components are equal.
- If two vectors are perpendicular, their dot product is zero.



2.4 Addition and Scalar Multiplication of Vectors

Let u and v be vector

1. The sum of u and v is

$$\mathbf{u}+\mathbf{v}=\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}+\begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}=\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n+v_n \end{bmatrix}$$

2. The scalar product of c and u is

$$cu=c\begin{bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_n \end{bmatrix}=\begin{bmatrix} cu_1 \\ cu_2 \\ \cdot \\ \cdot \\ \cdot \\ cu_n \end{bmatrix}$$

2.5 Eigenvalue and Eigenvectors

- For any $n \times n$ matrix A, there exists at least one number-vector pair $\boldsymbol{\lambda}, \mathbf{v}$ such that $A\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$

- Let A be an $n \times n$ matrix. A number λ is called an eigenvalue of A provided that there exists a nonzero vector in $\mathbb{R}^n$ such that $A\mathbf{v} = \boldsymbol{\lambda}\mathbf{v}$

- Every nonzero vector satisfying this equation is called an eigenvector of A corresponding to the eigenvalue $\boldsymbol{\lambda}$

- The zero vector is the trivial solution to the eigenvalue equation.

2.5 Example

Let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Find the eigenvalues and corresponding eigenvectors

Solution:

The number λ is an eigenvalue of A if there is a nonzero vector v such that $\mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix}$ such that 0 $\begin{bmatrix} x \\ y \end{bmatrix}$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} y \\ x \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

This matrix equation is equivalent to the system

$$\lambda x - y = 0$$
$$x - \lambda y = 0$$

Eliminating x we get

$(\lambda^2 - 1)\, y = 0$

2.5 Example

Either $\lambda = \pm 1$ *or* $y = 0$

$y = 0$ leads to the trivial solution $y = 0, x = 0$. Since eigenvector has to be nonzero vector, we get $\lambda = \pm 1$ Now for $\lambda_1 = 1$, we get

$$x - y = 0$$
$$x - y = 0$$

The solution set is $S = \left\{ \begin{bmatrix} t \\ t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

Similarly, for $\lambda_2 = -1$, the solution set is $S = \left\{ \begin{bmatrix} t \\ -t \end{bmatrix} \middle| \, t \in \mathbb{R} \right\}$

Specific eigenvectors can be found by specifying the value of t For t=1,

$$v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2.6 Characteristic Equation

If A is an $n \times n$ matrix, then

$$A\mathbf{v} = \lambda\mathbf{v}$$

For some number λ if and only if

$$A\mathbf{v} - \lambda\mathbf{v} = \mathbf{0} \text{ that is } (A - \lambda I)\mathbf{v} = A\mathbf{v} - \lambda I\mathbf{v} = 0$$

This gives us

$$\det A - \lambda I = 0$$

The number λ is an eigenvalue of the matrix A if and only if

$$\det A - \lambda I = 0$$

The equation $\det A - \lambda I = 0$ is called the **characteristic equation** of matrix A, and the expression $\det A - \lambda I$ is called the **characteristic polynomial** of A.

The set $V_\lambda = \{ v \epsilon \mathbb{R} \mid A\mathbf{v} = \lambda\mathbf{v} \}$
Is called the eigenspace of A corresponding to λ.

2.6 Example

Find the eigenvalues and corresponding eigenvectors of

$$A = \begin{bmatrix} 2 & -12 \\ 1 & -5 \end{bmatrix}$$

Give a description of the eigenspace corresponding to each eigenvalue.

Solution :
To find the eigenvalues we solve the characteristic equation

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -12 \\ 1 & -5-\lambda \end{vmatrix}$$
$$= (2-\lambda)(-5-\lambda) - (1)(-12)$$
$$= \lambda^2 + 3\lambda + 2$$
$$= (\lambda + 1)(\lambda + 2) = 0$$

2.6 Example

For λ=-1, we obtain the solution set $S = \left\{ \begin{bmatrix} 4t \\ t \end{bmatrix} \middle| t \epsilon \mathbb{R} \right\}$

Choosing t=1, we get the eigenvector $v_1 = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$

The eigenspace is $V_{\lambda_1} = \left\{ t \begin{bmatrix} 4 \\ 1 \end{bmatrix} \middle| t \text{ is any real number} \right\}$

Similarly we find that vector $v_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ s an eigenvector corresponding to λ2 = −2. The corresponding eigenspace is

$$V_{\lambda_2} = \left\{ t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \middle| t \text{ is any real number} \right\}$$