Lecture

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: SY BSc

Subject : Numerical methods and Algebra

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Chapter: Unit 4 Chapter 1

Chapter Name: Series and expansion

Today's Agenda

1. Sums and products
   1. Swapping the order of summation
   2 Product

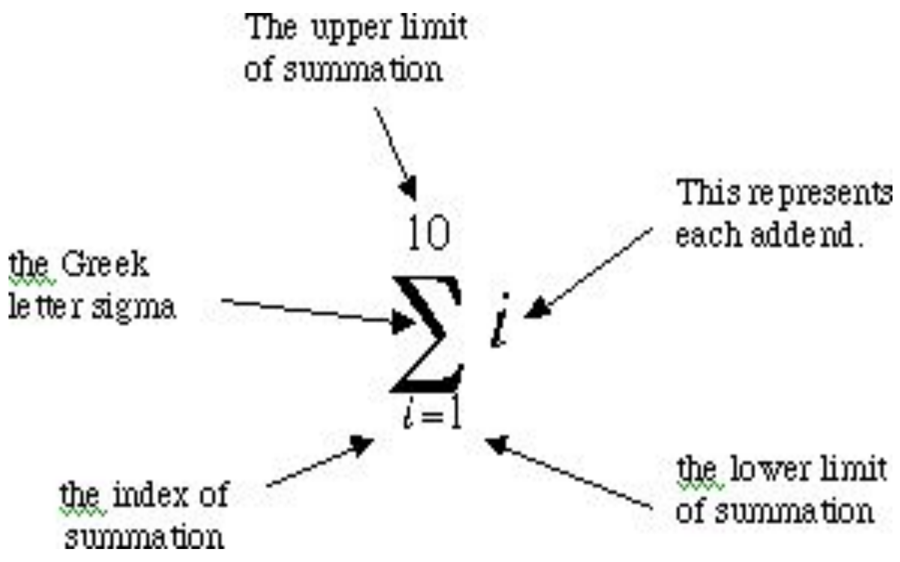
2. Arithmetic progression
   1. Properties of A. P

3. Geometric progression
   1. Notations
   2. Formulae
   3. Infinite geometric series
   4. Sum of infinite G. P

1 Sums And Products

SUMS

- Many statistical formulas involve repetitive summing operations. Consequently, we need a general notation for expressing such operations.
- We show how to write sums of numbers using the ∑ (sigma) notation as an abbreviation.
- They have the following general form $\sum_{i=1}^{N} x_i$



In the above expression, x are the numbers that are to be summed up, i is the summation index, 1 is the start value, N is the stop value.

1 Example

Question: Suppose our list has just 5 numbers, and they are 1,3,2,5,6.

Evaluate $\sum_{i=1}^{5} x_i^2$

Solution: In this case, we begin by setting i equal to 1, and evaluating x_1^2

Since $x_1 = 1$, our first evaluation produces a value of 1. Next, we set i equal to 2, and evaluate x_2^2 obtaining 9, which we add to the previous result of 1. We continue in this manner, obtaining $[1^2 + 3^2 + 2^2 + 5^2 + 6^2] = 75$

The order of operations is as important in summation expressions as in other mathematical notation.

In the following example, we compute the square of the sum of the numbers in our list.

1 Questions

Write the following using ∑ notation:

i) $3 + 3 + 3 + \ldots\ldots + 3 = 3n$

ii) $1^3 + 2^3 + 3^3 + \ldots\ldots + (n-1)^3$

iii) $2 + 5 + 10 + \ldots\ldots + 226$

iv) $\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + \cdots$

v) $-4 + 9 - 14 +$ ◊◊◊ $+ 99$

Solution

i) $\sum_{i=1}^{n} 3 = 3n$

ii) $\sum_{i=1}^{n-1} i^3$

iii) $\sum_{k=1}^{15} (k^2 + 1)$

iv) $\sum_{i=1}^{\infty} \frac{1}{2^{2i}}$

v) $\sum_{k=1}^{20} (-1)^k (5k - 1)$

1.1 Swapping the order of summation

When we have summation over two variables, we are able to swap the order of the summation. This may make the actual summation easier.

Suppose we have: $\sum_{x=1}^{10} \sum_{y=1}^{x} \cdots$

Then x takes the values between 1 and 10 inclusive, *ie* $1 \leq x \leq 10$. However, y takes the values between 1 and x . So inserting that in the correct position of the inequality, we would have $1 \leq y \leq x \leq 10$.

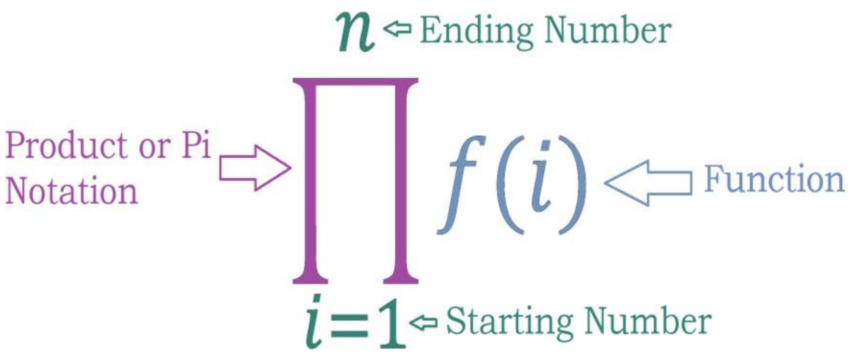
By swapping the order, we mean that instead of x summing between the extreme numbers we will let y take these values. Then by the position of x in the inequality it will be between y and 10.

Hence $\sum_{x=1}^{10} \sum_{y=1}^{x} \cdots = \sum_{y=1}^{10} \sum_{x=y}^{10} \cdots$

We have swapped the order of summation.

1.2 Product

- In the earlier course as we studied an abbreviation for summing numbers, now we study abbreviations to calculate product of a series of numbers.
- We show how to write products of numbers using the ∏ (pi) notation as an abbreviation.
- It has the general form like;



1.2 Example

Question: Suppose our list has the following numbers 1,2,3,4,5,6. Find the product of the squares of these numbers.

Solution: Writing this using the product notation,

$$\prod_{i=1}^{6} i^2 = (1)(4)(9)(16)(25)(36)$$

$$= 518400$$

1.2 Question

Write the following series using $\prod$ notation

i) $3 \times 5 \times 7 \times \dots\dots \times (2n+1)$

ii) $e \times e^2 \times e^3 \times \dots\dots \times e^{2n+3}$

iii) $\frac{1}{4\alpha} \times \frac{1}{9\alpha^2} \times \frac{1}{16\alpha^3} \times \cdots \times \frac{1}{256\alpha^{15}}$

Solutions

(i) $$\prod_{i=1}^{n}(2i+1)$$

(ii) $$\prod_{i=1}^{2n+3} e^{i}$$

(iii) $$\prod_{i=1}^{15}\frac{1}{(i+1)^2\alpha^{i}}$$

2 Arithmetic Progression (A.P.)

- In mathematics, an arithmetic progression (AP) is a sequence of numbers such that the difference between the consecutive terms is constant. Difference here means the second minus the first. For instance, the sequence 5, 7, 9, 11, 13, 15, . . . is an arithmetic progression with *common difference* of 2.

- **Nth term of A.P**: Since the numbers go up by the same amount each time we can use this to determine future terms. In general using *a* to stand for the first term and d for the common difference then the *n*th term is = $a + (n - 1)d$.

- **Sum of A.P**: We can find the sum of the n terms of an arithmetic progression using the formula:

$$S_n = \frac{1}{2}n\left[2a + (n-1)d\right]$$

2.1 Properties of an A.P.

- If a constant number is added or subtracted from each term of an A.P, then the resulting term in the sequence are also in A.P with the same common difference.
- If each term in an A.P is divided or multiply with a constant non-zero number, then the resulting sequence is also in an A.P
- Three number x, y and z are in an A.P if $2y = x + z$
- A sequence is an A.P if its n^{th} term is a linear expression.
- If we select terms in the regular interval from an A.P, these selected terms will also be in AP.
- If the difference between all the consecutive terms of a sequence is the same, then the sequence is an A.P.

2.1 Examples

1) Is the row 1,11,21,31... an arithmetic progression?

Solution:

Yes, it is an arithmetic progression. Its first term is 1 and the common difference is 10.

2) Find the sum of the first 10 numbers of this arithmetic series: 1, 11, 21, 31...

Solution:

we can use this formula

$S = {}^1/_2(2a_1 + d(n-1))n$

$S = {}^1/_2(2x1 + 10(10-1))10$

$= 5(2 + 90)$

$= 5.92$

2.1 Examples

3) A clock strikes the number of times of the hour. How many strikes does it make in one day?

Solution:

For the first 12 hours of the day, the clock will strike 1+ 2+ 3+ .....+ 12 times

$$= 12(1+12)/2 = 78 \text{ times}$$

For the next 12 hours, there will be another 78 times, so in one day, the clock will strike 156 times.

2.1 Question

A piece of equipment cost a certain factory 600000. If it depreciates in value 15% the first year, 13.5 % the next year, 12% the third year, and so on, what will be its value at the end of 10 years, all percentages applying to the original cost?

Solution

Let the cost of an equipment be Rs. 100.

Now the percentages of depreciation at the end of 1st, 2nd, 3rd years are 15, 13.5, 12, which are in A.P., with $a = 15$ and $d = -1.5$.

Hence, percentage of depreciation in the tenth year = $a + (10-1)\,d$

$$= 15 + 9\,(-1.5) = 1.5$$

Also total value depreciated in 10 years = $15 + 13.5 + 12 + \ldots + 1.5 = 82.5$
Hence, the value of equipment at the end of 10 years=$100 - 82.5 = 17.5$.

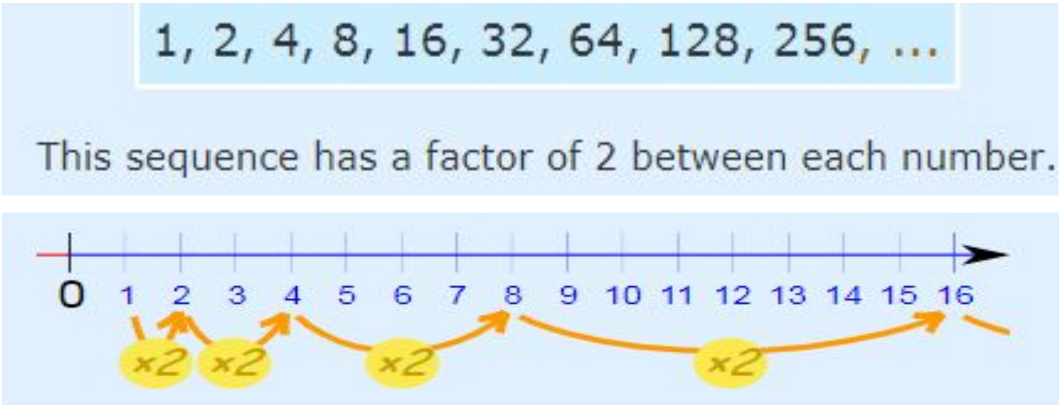
The total cost being Rs. 6,00,000/100 * 17.5 = Rs. 1,05,000.

3 Geometric Progression (G.P.)

A Geometric Progression (GP) is formed by **multiplying** a starting number (a_1) by a number r, called the **common ratio.**

Eg The progression 5,10,20,40,80,160, has first term $a1$=5, and common ratio r=2.

In this example, we started with 5 and multiplied by 2 each time to get the next number in the progression.



3.1 Notations

The first term is

$$a_1$$

The second term is obtained by multiplying the first by r

$$a_1 r$$

The third term is obtained by multiplying the second by r

$$a_1 r^2$$

The fourth term is obtained by multiplying the third by r

$$a_1 r^3$$

We continue this pattern and can see that in general, the n-th term is

$$a_1 r^{n-1}$$

3.2 Formulae

- The *n*–th term of a geometric progression is given by:

$$\boxed{a_n = a_1 r^{n-1}}$$

- The sum to *n* terms of a GP means:

$$a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \ldots + a_1 r^{n\text{-}1}$$

This can be solved using the formulae;

$$\boxed{S_n = \frac{a_1(1 - r^n)}{1 - r} \quad (r \neq 1)}$$

3.2 Example 1

Questions: Find the 50^{th} term of the geometric progression 5, 10, 20, 40, 80, …

Solution:

Since $a_1 = 5$, $r = 2$, and using

$$a_n = a_1 r^{n\text{-}1},$$

we have:

$$a_{50} = (5)(2^{50-1})$$

$$= 2{,}814{,}749{,}767{,}106{,}560$$

$$\approx 2.81 \times 10^{15}$$

3.3 Infinite Geometric Series

In the last section we studied geometric progression in the finite space. Now let's have a look at the infinite geometric progression.

An infinite geometric series is the sum of an infinite geometric sequence . This series would have no last term.

The general form of the infinite geometric series is $a1+a1r+a1r^2+a1r^3+......$, where a1 is the first term and r is the common ratio.

EG. $-\frac{5}{4}, \frac{5}{16}, -\frac{5}{64}, \frac{5}{256}, ...$

This is an infinite geometric series.

3.4 Sum of an infinite G.P.

- We can find the sum of all finite geometric series.
- But in the case of an infinite geometric series when the common ratio is greater than one, the terms in the sequence will get larger and larger and if you add the larger numbers, you won't get a final answer. The only possible answer would be infinity. So, we don't deal with the common ratio greater than one for an infinite geometric series.
- If the common ratio r lies between −1−1 to 11 , we can have the sum of an infinite geometric series. That is, the sum exits for | r |<1| r |<1 .
- The sum S of an infinite geometric series with −1<r<1−1<r<1 is given by the formula,

- An infinite series that has a sum is called a convergent series and the sum Sn is called the partial sum of the series.

3.4 Example

Question: work out the following sum

$$\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r$$

Solution:

$$\sum_{r=1}^{\infty}\left(\frac{1}{3}\right)^r = \left(\frac{1}{3}\right)^1 + \left(\frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3 + \ldots + \left(\frac{1}{3}\right)^n + \ldots$$ expand the given sum

$$= \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \ldots + \left(\frac{1}{3}\right)^n + \ldots$$

$$r = \frac{a_2}{a_1} = \frac{\frac{1}{9}}{\frac{1}{3}} = \frac{1}{3}$$ solve for r using $r = \frac{a_2}{a_1}$

$$S_\infty = \frac{\frac{1}{3}}{1-\frac{1}{3}}$$ substitute $a_1 = \frac{1}{3}$ and r $= \frac{1}{3}$ to $S_\infty = \frac{a_1}{1-r}$

$$= \frac{\frac{1}{3}}{\frac{2}{3}}$$ subtract $\frac{1}{3}$ from *1*

$$S_\infty = \frac{1}{2}$$ divide $\frac{1}{3}$ by $\frac{2}{3}$