IAQS INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: **Numerical Methods and Algebra**

Chapter:

Category: **Assignment**

Assignment 1

Numerical Methods and Algebra

1. Find the value of $\sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$

Solution

Let $x = \sqrt{7+\sqrt{7+\sqrt{7+\cdots}}}$

Since the quantity inside the square root sign is a never ending, repeating sequence, we can make the following substitution

$$x = \sqrt{7+\left(\sqrt{7+\sqrt{7+\cdots}}\right)}$$

$$x = \sqrt{7+x}$$

Squaring both sides we get

$$x^2 = 7 + x$$

$$x^2 - x - 7 = 0$$

We cannot factorise so we use the determinant formula

$$x = \frac{1 \pm \sqrt{1+4\times 7}}{2} = \frac{1}{2} \pm \frac{\sqrt{29}}{2}$$

2. Find the value of $\frac{x}{y}$ in the following equation:

$$7x^2 - 6y^2 = -11x^2 - 4y^2$$

Diving both sides by y^2 we get

$$\frac{7x^2}{y^2} - \frac{6y^2}{y^2} = -\frac{11x^2}{y^2} - \frac{4y^2}{y^2}$$

$$7\left(\frac{x}{y}\right)^2 - 6 = -11\left(\frac{x}{y}\right)^2 - 4$$

Solution

Let $\frac{x}{y}$ be t

We get

$$7t^2 - 6 = -11t^2 - 4$$

$$18t^2 = 2$$

$$t^2 = \frac{2}{18} = \frac{1}{9}$$

$$t = \pm\frac{1}{3}$$

$$\frac{x}{y} = \frac{1}{3} or -\frac{1}{3}$$

3. What ordered pairs satisfy the system

$$x + 3y = 4$$
$$x^2 + y^2 - 4y = 12$$

Solution

From the first equation, we get $x = 4 - 3y$

Substituting in the second equation, we get

$$(4 - 3y)^2 + y^2 - 4y = 12$$
$$16 + 9y^2 - 24y + y^2 - 4y - 12 = 0$$
$$10y^2 - 28y + 4 = 0$$
$$5y^2 - 14y + 2 = 0$$
$$y = \frac{14 \pm \sqrt{196 - 40}}{10} = \frac{14 \pm 2\sqrt{39}}{10} = \frac{7 \pm \sqrt{39}}{5}$$

For $y = \frac{7+\sqrt{39}}{5}, x = 4 - 3\left(\frac{7+\sqrt{39}}{5}\right) = \frac{-1-3\sqrt{39}}{5}$

For $y = \frac{7-\sqrt{39}}{5}, x = 4 - 3\left(\frac{7-\sqrt{39}}{5}\right) = \frac{-1+3\sqrt{39}}{5}$

The ordered pairs of x,y are $\left(\frac{-1-3\sqrt{39}}{5}, \frac{7+\sqrt{39}}{5}\right)$ and $\left(\frac{-1+3\sqrt{39}}{5}, \frac{7-\sqrt{39}}{5}\right)$

4. The quadratic function defined by the equation $d = 2r^2 - 16r + 34$ gives the density of smoke, d, in millions of particles per cubic foot for a certain type of diesel engine. The input variable, r, represents the speed of the engine in hundreds of revolutions per minute
    a. Determine the density of smoke when r = 3.5 (350 revolutions per minute).
    b. Determine the number of revolutions per minute for minimum smoke. What is the minimum output?
    c. If the density of smoke is determined to be 100 million particles per cubic foot, determine the speed of the engine.

Solution

$$d = 2r^2 - 16r + 34$$

a. For r=3.5, we get

$$d = 2(3.5)^2 - 16(3.5) + 34 = 2.5$$

b. $d = 2r^2 - 16r + 34$

$$= 2(r^2 - 8r + 17)$$

Using completion of squares

$$= 2(r^2 - 8r + 16 - 16 + 17)$$
$$= 2((r-4)^2 + 1)$$

For d to be minimum, the quantity inside the brackets has to be minimized

Since $(r-4)^2$ is non-negative for all real values of r, the minimum value is zero for r=4

The number of revolutions per minute for minimum smoke is r=4 or 400 revolutions per minute

The corresponding amount of smoke = 2 millions of particles per cubic feet

c. For d=100

$$100 = 2r^2 - 16r + 34$$
$$2r^2 - 16r - 66 = 0$$
$$r^2 - 8r - 33 = 0$$
$$r^2 - 11r + 3r - 33 = 0$$
$$r(r-11) + 3(r-11) = 0$$
$$(r+3)(r-11) = 0$$

Since r can not be negative, r=11

5. You contact two car rental companies and obtained the following information for the 1-day cost of renting a car.
   Company 1: Rs 1500 per day plus Rs 25 per km
   Company 2: Rs 2100 per day plus Rs 22 per km
   Let n represent the total number of km driven in 1 day.
   a. Write an expression to determine the total cost, C, of renting a car for 1 day from company 1.
   b. Write an expression to determine the total cost, C, of renting a car for 1 day from company 2.
   c. Use the expressions in parts a and b to write an inequality that can be used to determine for what number of km it is less expensive to rent the car from company 2.
   d. Solve the inequality.

Solution

a. $C = 1500 + 25n$

b. $C = 2100 + 22n$

c. For the given condition,

$$2100 + 22n < 1500 + 25n$$

d. $3n > 2100 - 1500$

$$3n > 600$$
$$n > 200$$

Company 2 is cheaper for trips of more than 200 km

6. Use the Bisection method to find solutions accurate to within 10^{-2} for $x^3 - 7x^2 + 14x - 6 = 0$ on each interval

a. [0, 1]

iteration	a	b	pi	f(a)	f(b)	f(pi)
1	0	1	0.50	-6	2	-0.625
2	0.5	1	0.75	-0.625	2	0.984375
3	0.5	0.75	0.63	-0.625	0.984375	0.259766
4	0.5	0.625	0.56	-0.625	0.259766	-0.16187
5	0.5625	0.625	0.59	-0.16187	0.259766	0.054047
6	0.5625	0.59375	0.58	-0.16187	0.054047	-0.05262
7	0.578125	0.59375	0.59	-0.05262	0.054047	0.001031
8	0.578125	0.585938	0.58	-0.05262	0.001031	-0.02572
9	0.582031	0.585938	0.58	-0.02572	0.001031	-0.01232

Solution=0.58

b. [1, 3.2]

iteration	a	b	pi	f(a)	f(b)	f(pi)
1	1	3.2	2.10	2	-0.112	1.791
2	2.1	3.2	2.65	1.791	-0.112	0.552125
3	2.65	3.2	2.93	0.552125	-0.112	0.085828
4	2.925	3.2	3.06	0.085828	-0.112	-0.05444
5	2.925	3.0625	2.99	0.085828	-0.05444	0.006328
6	2.99375	3.0625	3.03	0.006328	-0.05444	-0.02652
7	2.99375	3.028125	3.01	0.006328	-0.02652	-0.0107
8	2.99375	3.010938	3.00	0.006328	-0.0107	-0.00233
9	2.99375	3.002344	3.00	0.006328	-0.00233	0.001961

Solution = 3.00

c. [3.2, 4]

iteration	a	b	pi	f(a)	f(b)	f(pi)
1	3.2	4	3.60	-0.112	2	0.336
2	3.2	3.6	3.40	-0.112	0.336	-0.016
3	3.4	3.6	3.50	-0.016	0.336	0.125
4	3.4	3.5	3.45	-0.016	0.125	0.046125

5	3.4	3.45	3.43	-0.016	0.046125	0.013016
6	3.4	3.425	3.41	-0.016	0.013016	-0.002
7	3.4125	3.425	3.42	-0.002	0.013016	0.005382
8	3.4125	3.41875	3.42	-0.002	0.005382	0.00166

Solution = 3.42

7. The fourth-degree polynomial

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$

has two real zeros, one in [−1, 0] and the other in [0, 1]. Attempt to approximate these zeros to within 10^{-4} using the Newton's method.

Solution

$$f(x) = 230x^4 + 18x^3 + 9x^2 - 221x - 9$$
$$f'(x) = 920x^3 + 54x^2 + 18x - 221$$

For interval [-1,0]
Let us take $p_1 = 0.5$

$$p_2 = -0.1505$$
$$p_3 = -0.0418$$
$$p_4 = -0.0407$$
$$p_5 = -0.0407$$

Solution = -0.0407

For interval [0,1]
Let us take $p_1 = 0.85$

$$p_2 = 0.9987$$
$$p_3 = 0.9648$$
$$p_4 = 0.9624$$
$$p_5 = 0.9624$$

Solution=0.9624

8. You are enrolled in an algebra course at your college. You achieved grades of 70, 86, 81, and 83 on the first four exams. The final exam counts the same as an exam given during the semester.
   a. If x represents the grade on the final exam, write an expression that represents your course average (arithmetic mean).

b. If your average is greater than or equal to 80 and less than 90, you will earn a B in the course. Using the expression from part a for your course average, write a compound inequality that must be satisfied to earn a B.

c. Solve the inequality.

Solution

a.

$$course\ average = \frac{70 + 86 + 81 + 83 + x}{5} = 64 + 0.2x$$

b. $80 \le (64 + 0.2x) \le 90$

c. $80 \le (64 + 0.2x) \le 90$

$$16 \le 0.2x \le 26$$
$$80 \le x \le 130$$

Since x cannot be greater than 100, solution is

$$80 \le x \le 100$$

9. The NPV for a project at a discount rate of 4% is 8.54. The NPV at a discount rate of 20% is -11.81. Using the above data points, find the IRR using linear interpolation.

Solution

IRR is the discount rate for which NPV = 0

We assume a linear relationship between the discount rate and the NPV for discount rate between 4% and 20%

So we can tabulate the values as

Discount rate	NPV
4%	8.54
?	0
20%	-11.81

Let the IRR be x

$$\frac{8.54 - 0}{4 - x} = \frac{8.54 - (-11.81)}{4 - 20}$$

$$\frac{8.54}{4 - x} = \frac{20.35}{-16}$$

$$x - 4 = 16 \times \frac{8.54}{20.35}$$

$$x = 4 + 6.7145 = 10.71\%$$

The IRR is 10.71%

10. The makers of a new food delivery app estimate that with x thousand orders their monthly revenue and cost (in lakhs of Rs) are given by the following:

$$R(X) = 32x - 0.21x^2$$
$$C(x) = 195 + 12x$$

Determine the number of orders needed for the app to remain profitable.

Solution

In order to remain profitable, revenue must be greater than cost

Or

$$R(x) > C(x)$$
$$32x - 0.21x^2 > 195 + 12x$$
$$-0.21x^2 + 20x - 195 > 0$$
$$0.21x^2 - 20x + 195 < 0$$

Solving $0.21x^2 - 20x + 195 = 0$

$$x = \frac{20 \pm \sqrt{400 - 4 \times 0.21 \times 195}}{0.42} = \frac{20 \pm \sqrt{400 - 79.8)}}{0.42} = \frac{20 \pm 17.89}{0.42}$$
$$The\ roots\ are\ 90.214\ and\ 5.024$$

Therefore, the solution to the inequality is

$$5.024 < x < 90.214$$

The app remains profitable if the monthly orders remain between 5024 and 90214.