INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject : Numerical Methods

Matrices

Matrix

- A set of numbers, arranged in rows and columns so as to forma a rectangular array
- The numbers are called the elements, or entries, of the matrix.
- Example:

$$\begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

This matrix has 2 rows and 3 columns. It is a 2x3 matrix.

Types of Matrices

1. **Row matrix**: A row matrix has only one row

E.g. $\begin{bmatrix} 1 & 2 \end{bmatrix}$

2. **Column matrix**: A column matrix has only one column.

E.g. $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ A column matrix is also called a vector.

3. **Square matrix:** A square matrix has an equal number of rows and columns

E.g. $\begin{bmatrix} 1 & 2 & -2 \\ 8 & 2 & 2 \\ 1 & 2 & -1 \end{bmatrix}$ (A 3x3 square matrix)

Types of Matrices

4. Diagonal matrix: A diagonal matrix has non-zero diagonal elements and all other elements are zero

E.g. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

5. Scalar Matrix: A scalar matrix has all main diagonal entries the same, with zero everywhere else:

E.g. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$

Types of Matrices

6. Triangular matrix:

a. Lower Triangular matrix is a matrix where all entries above the main diagonal are zero E.g. $\begin{bmatrix} 4 & 0 & 0 \\ -2 & 3 & 0 \\ 5 & 1 & 2 \end{bmatrix}$

b. Upper Triangular matrix is a matrix where all entries below the main diagonal are zero E.g. $\begin{bmatrix} 4 & 2 & 1 \\ 0 & 3 & 3 \\ 0 & 0 & 2 \end{bmatrix}$

Types of Matrices

7. Null matrix: A null matrix has all of its elements as zero.

8. Identity Matrix: An identity matrix is a square matrix that has 1s on the main diagonal and 0s everywhere else.

- Its symbol is I
- It is the matrix equivalent of the number 1; multiplying a matrix with the identity matrix keeps the original matrix unchanged:

$$A \times I = A$$
$$I \times A = A$$
$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This is a 3 x 3 identity matrix

Matrix Notation

- A matrix is usually denoted by a capital letter (such as A or B)
- Each entry (or "element") is shown by a lower case letter with a "subscript" of row,column
- Example:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

Matrix Addition

- To add two matrices, add the numbers in the matching positions
- The two matrices must have the same size, i.e. the rows must match in size and the columns must also match in size.
- Example

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 3+4 & 8+0 \\ 4+1 & 6-9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

Matrix Addition & Subtraction

- For matrix **addition or subtraction**, the two matrices must have the same size, i.e. the rows must match in size and the columns must also match in size.
- To **add** two matrices, add the numbers in the matching positions
- Example

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 3+4 & 8+0 \\ 4+1 & 6-9 \end{bmatrix} = \begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix}$$

- To subtract two matrices: subtract the numbers in the matching positions: subtracting is actually defined as the addition of a negative matrix: A + (−B).
- Example

$$\begin{bmatrix} 3 & 8 \\ 4 & 6 \end{bmatrix} - \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 3-4 & 8-0 \\ 4-1 & 6-(-9) \end{bmatrix} = \begin{bmatrix} -1 & 8 \\ 3 & 15 \end{bmatrix}$$

Matrix Transpose

- To "transpose" a matrix, swap the rows and columns.
- We put a "T" in the top right-hand corner to mean transpose
- Example

$$A = \begin{bmatrix} 6 & 4 & 24 \\ 1 & -9 & 8 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 6 & 1 \\ 4 & 19 \\ 24 & 8 \end{bmatrix}$$

An square matrix A is **symmetric** provided $A^T = A$

Multiplication by a Constant

- We can multiply a matrix by a constant
- We call the constant a scalar, so officially this is called "scalar multiplication".
- Example:

$$2 \times \begin{bmatrix} 4 & 0 \\ 1 & -9 \end{bmatrix} = \begin{bmatrix} 2 \times 4 & 2 \times 0 \\ 2 \times 1 & 2 \times -9 \end{bmatrix} = \begin{bmatrix} 8 & 0 \\ 2 & -18 \end{bmatrix}$$

Matrix Multiplication

- Matrix Multiplication is an operation that depends on the order of matrices.
- The **number of columns of the 1st matrix must equal the number of rows of the 2nd matrix.** $A \times B$ is defined only if, no of columns in A=no of rows in B
- And the result will have the same number of rows as the 1st matrix, and the same number of columns as the 2nd matrix.
- If A is a $m \times n$ matrix, B is a $n \times p$ matrix, then $A \times B$ is a $m \times p$ matrix
- Matrix multiplication is not commutative. i.e. when we change the order of multiplication, the answer changes.

$$AB \neq BA$$

Matrix Multiplication

- Matrix Multiplication is row by column
- If A is a 2×3 matrix, B is a 3×2 matrix, then C is a 2×2 matrix

$$c_{ik} = \sum_{j=1}^{3} a_{ij} b_{jk}$$

So,

$$\begin{aligned} c_{11} &= a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} \\ c_{12} &= a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ c_{21} &= a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} \\ c_{22} &= a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{aligned}$$

Example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

$$A \times B = \begin{bmatrix} 1 \times 7 + 2 \times 9 + 3 \times 11 & 1 \times 8 + 2 \times 10 + 3 \times 12 \\ 4 \times 7 + 5 \times 9 + 6 \times 11 & 4 \times 8 + 5 \times 10 + 6 \times 12 \end{bmatrix}$$

$$= \begin{bmatrix} 7 + 18 + 33 & 8 + 20 + 36 \\ 28 + 45 + 66 & 32 + 50 + 72 \end{bmatrix}$$

$$= \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

Properties of Matrix Addition and Scalar Multiplication

Let $A, B,$ and C be $m \times n$ matrices and c and d be real numbers.

1. $A + B = B + A$
2. $A + (B + C) = (A + B) + C$
3. $c(A + B) = cA + cB$
4. $(c + d)A = cA + dA$
5. $c(dA) = (cd)A$
6. The $m \times n$ matrix with all zero entries, denoted by **0,** is such that $A + \mathbf{0} = \mathbf{0} + \boldsymbol{A} = \boldsymbol{A}$.
7. For any matrix A, the matrix $-A$, whose components are the negative of each component of A, is such that $A + (-A) = (-A) + A = \mathbf{0}$

Properties of Matrix Multiplication

Let $A, B,$ and C be matrices with sizes so that the given expressions are all defined, and let c be a real numbers.

1. $A(BC) = (AB)C$
2. $c(AB) = (cA)B = A(cB)$
3. $A(B + C) = AB + AC$
4. $(B + C)A = BA + CA$

Inverse of a Matrix

- The Inverse of a Matrix is the same idea as the reciprocal of a number but we write it A^{-1}
- When we multiply a matrix by its inverse we get the Identity Matrix (which is like "1" for matrices):

$$A \times A^{-1} = I$$

- Same thing when the inverse comes first: $A^{-1} \times A = I$
- **Definition**: The inverse of A is A^{-1} only when:

$$A \times A^{-1} = A^{-1} \times A = I$$

- Sometimes there is no inverse at all.

Why we need an Inverse

With matrices we don't divide as there is no concept of dividing by a matrix. But we can multiply by an inverse, which achieves the same thing.

Say we want to find matrix X, and we know matrix A and B:

$$XA = B$$

It would be nice to divide both sides by A (to get $X = B/A$), but remember we can't divide. But what if we multiply both sides by A^{-1} ?

$$XAA^{-1} = BA^{-1}$$

And we know that $AA^{-1} = I$, so:

$$XI = BA^{-1}$$

We can remove I (for the same reason we can remove "1" from 1x = ab for numbers):

$$X = BA^{-1}$$

And we have our answer (assuming we can calculate A^{-1}).

- Note: Order of multiplication is important.

The Inverse May not Exist

- The inverse of a matrix , if it exists, is unique
- When the inverse of a matrix A exists, we call A **Invertible**. Otherwise the matrix is called **noninvertible**.
- For a 2×2 matrix A=$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, A is invertible if and only if $\boldsymbol{ad - bc \neq 0}$

Example

Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

In order for a 2×2 matrix $B = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ to be an inverse of A, B must satisfy

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} = \begin{bmatrix} x_1 + x_3 & x_2 + x_4 \\ x_1 + 2x_3 & 2x_2 + 2x_4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix equation is equivalent to the system

$$\begin{aligned} x_1 \quad + x_3 \quad &= 1 \\ x_2 \quad + x_4 &= 0 \\ x_1 \quad + 2x_3 \quad &= 0 \\ x_2 \quad + 2x_4 &= 1 \end{aligned}$$

Thus the solution is $x_1 = 2, x_2 = -1, x_3 = -1, x_4 = 1$, and the inverse matrix is

$$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}$$

Example 2

Find the inverse of the matrix: $A = \begin{bmatrix} 1 & 1 & -2 \\ -1 & 2 & 0 \\ 0 & -1 & 1 \end{bmatrix}$

To find the inverse we place the identity matrix on the right to form the 3×6 matrix

$$\begin{bmatrix} 1 & 1 & -2 & | & 1 & 0 & 0 \\ -1 & 2 & 0 & | & 0 & 1 & 0 \\ 0 & -1 & 1 & | & 0 & 0 & 1 \end{bmatrix}$$

Now use row operations to reduce the matrix on the left to the identity matrix while performing the same operations to the matrix on the right.

The final result is $\begin{bmatrix} 1 & 0 & 0 & | & 2 & 1 & 4 \\ 0 & 1 & 0 & | & 1 & 1 & 2 \\ 0 & 0 & 1 & | & 1 & 1 & 3 \end{bmatrix}$

So the inverse matrix is $A^{-1} = \begin{bmatrix} 2 & 1 & 4 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{bmatrix}$

System of Linear Equations

- A system of m linear equations in n variables, or a linear system is a collection of equations of the form:

$$
\begin{array}{ccccccccc}
a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\
a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\
a_{31}x_1 & + & a_{32}x_2 & + & \cdots & + & a_{3n}x_n & = & b_3 \\
. & & . & & . & & . & & . \\
. & & . & & . & & . & & . \\
. & & . & & . & & . & & . \\
a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m
\end{array}
$$

- This is also referred to as an $m \times n$ linear system

Example

$$\begin{aligned} -2x_1 + 3x_2 + x_3 - x_4 &= -2 \\ x_1 \qquad\qquad + x_3 - 4x_4 &= 1 \\ 3x_1 - x_2 \qquad\qquad - x_4 &= 3 \end{aligned}$$

Is a linear system of three equations in four variables, or a 3×4 linear system.

Solution to a System of Linear Equations

- A solution to a linear system with n variables is an ordered sequence $(s_1, s_2, \ldots, s_n)$ such that each equation is satisfied for $x_1 = s_1, x_2 = s_2, \ldots . x_n = s_n$.
- The **general solution** or **solution set** is the set of all possible solutions.
- A $m \times n$ linear system has either
  - a unique solution (consistent system),
  - infinitely many solutions (consistent system), or
  - no solution (inconsistent system)

Triangular Form of a Linear System

- An $m \times n$ linear system is in triangular form provided that the coefficients $a_{ij} = 0$ whenever $i > j$.
- In this case we refer to the linear system as a **triangular system**.
- Example:

$$\begin{aligned} x_1 - 2x_2 + x_3 &= -1 \\ x_2 - 3x_3 &= 5 \\ x_3 &= 2 \end{aligned}$$

- When a linear system is in triangular form, then the solution set can be obtained using **back substitution.**
- In the above triangular system, we can see that $x_3 = 2.$ Substituting this in the second equation, we obtain $x_2 - 3(2) = 5$, so $x_2 = 11$. Finally , using these values in the first equation, we have $x_1 - 2(11) + 2 = -1$, so $x_1 = 19$.
- The solution is written as (**19,11,2)**

Equivalent Systems

- Two linear systems are **equivalent** if they have the **same solutions**.

Performing any of the following operations on a linear system produces an equivalent linear system:

1. **Interchanging any two equations**
2. **Multiplying any equation by a nonzero constant**
3. **Adding a multiple of one equation to another.**

Example

- Solve

$$\begin{aligned} x + y + z &= 4 \\ -x - y + z &= -2 \\ 2x - y + 2z &= 2 \end{aligned}$$

To convert the system into an equivalent triangular system, we first eliminate the variable x in the second and third equations to obtain

$$\begin{aligned} x + y + z &= 4 \\ -x - y + z &= -2 \\ 2x - y + 2z &= 2 \end{aligned} \qquad \begin{aligned} E_1 + E_2 &\to E_2 \\ -2E_1 + E_3 &\to E_3 \end{aligned} \qquad \longrightarrow \qquad \begin{aligned} x + y + z &= 4 \\ 2z &= 2 \\ -3y \qquad &= -6 \end{aligned}$$

Interchanging the second and third equations gives the triangular linear system

$$\begin{aligned} x + y + z &= 4 \\ 2z &= 2 \\ -3y \qquad &= -6 \end{aligned} \qquad E_2 \leftrightarrow E_3 \qquad \longrightarrow \qquad \begin{aligned} x + y + z &= 4 \\ -3y \qquad &= -6 \\ 2z &= 2 \end{aligned}$$

Using back substitution, we have$z = 1, y = 2, and\ x = 4 - y - z = 1.$

Augmented Matrix

- Solving a linear system, by elimination method requires only the coefficients of the variables and the constants on the right-hand side
- The coefficients and the constants can be recorded by using columns as placeholders for variables.

$$\begin{aligned} -4x_1 + 2x_2 \phantom{{}-4x_3} - 3x_4 &= 11 \\ 2x_1 - x_2 - 4x_3 + 2x_4 &= -3 \\ 3x_2 \phantom{{}-4x_3} - x_4 &= 0 \\ -2x_1 \phantom{{}+2x_2 - 4x_3} + x_4 &= 4 \end{aligned}$$

Can be recorded in matrix form as →

$$\left[\begin{array}{cccc|c} -4 & 2 & 0 & -3 & 11 \\ 2 & -1 & -4 & 1 & -3 \\ 0 & 3 & 0 & -1 & 0 \\ -2 & 0 & 0 & 1 & 4 \end{array}\right]$$

- This matrix is called the **augmented matrix** of the linear system
- The augmented matrix with the last column deleted is called the **coefficient matrix**

Example

Linear System

$$
\begin{aligned}
x + y - z &= 1 \\
2x - y + z &= -1 \\
-x - y + 3z &= 2
\end{aligned}
$$

Using the operations $-2E_1 + E_2 \to E_2$ and $E_1 + E_3 \to E_3$, we obtain the equivalent triangular system:

$$
\begin{aligned}
x + y - z &= 1 \\
-3y + 3z &= -3 \\
2z &= 3
\end{aligned}
$$

Corresponding Augmented Matrix

$$
\begin{bmatrix} 1 & 1 & -1 & | & 1 \\ 2 & -1 & 1 & | & -1 \\ -1 & -1 & 3 & | & 2 \end{bmatrix}
$$

Using the operations $-2R_1 + R_2 \to R_2$ and $R_1 + R_3 \to R_3$, we obtain the equivalent triangular system:

$$
\begin{bmatrix} 1 & 1 & -1 & | & 1 \\ 0 & -3 & 3 & | & -3 \\ 0 & 0 & 2 & | & 3 \end{bmatrix}
$$

Operations on Augmented Matrix

- Any of the following operations performed on an augmented matrix , corresponding to a linear system, produces an augmented matrix corresponding to an equivalent linear system

1. Interchanging any two rows
2. Multiplying any row by a nonzero constant
3. Adding a multiple of one row to another

Solving Linear Systems with Augmented Matrix

1. Write the augmented matrix of the linear system
2. Use row operations to reduce the augmented matrix to triangular form
3. Interpret the final matrix as a linear system (which is equivalent to the original)
4. Use back substitution to obtain the solution

Example

Write the augmented matrix and solve the linear system

$$x - 6y - 4z = -5$$
$$2x - 10y - 9z = -4$$
$$-x + 6y + 5z = 3$$

Example- solution

To solve this system, we write augmented matrix:

$$\begin{bmatrix} 1 & -6 & -4 & | & -5 \\ 2 & -10 & -9 & | & -4 \\ -1 & 6 & 5 & | & 3 \end{bmatrix}$$

The augmented matrix is reduced to triangular form as follows:

$$\begin{bmatrix} 1 & -6 & -4 & | & -5 \\ 2 & -10 & -9 & | & -4 \\ -1 & 6 & 5 & | & 3 \end{bmatrix} \qquad \begin{matrix} -2R_1 + R_2 \rightarrow R_2 \\ R_1 + R_3 \rightarrow R_3 \\ \ \end{matrix} \qquad \begin{bmatrix} 1 & -6 & -4 & | & -5 \\ 0 & 2 & -1 & | & 6 \\ 0 & 0 & 1 & | & -2 \end{bmatrix}$$

The equivalent triangular linear system is:

$$\begin{aligned} x - 6y - 4 &= -5 \\ 2y - z &= 6 \\ z &= -2 \end{aligned}$$

Which has the solution $x = -1, y = 2, and\ z = -2$

Matrix Equations -1

Consider the linear system

$$x - 6y - 4z = -5$$
$$2x - 10y - 9z = -4$$
$$-x + 6y + 5z = 3$$

The matrix of coefficients is given by

$$A = \begin{bmatrix} 1 & -6 & -4 \\ 2 & -10 & -4 \\ -1 & 6 & 5 \end{bmatrix}$$

Let **x** and **b** be the vectors

$$\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \; and \; \mathbf{b} = \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix}$$

Thus the original system can be written as

$$A\mathbf{x} = \mathbf{b}$$

This is the **matrix form** of the linear system and **x** is the vector form of the solution

Matrix Equations - 2

If A is invertible, we have

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

Since matrix multiplication is associative: $(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$

Therefore

$$\mathbf{x} = A^{-1}b$$

$$A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, the solution to the linear system in vector form is given by

$$x = A^{-1}b = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

That is $x = -1, y = 2 \; and \; z = -2$

Matrix Equations - 2

If A is invertible, we have

$$A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$$

Since matrix multiplication is associative: $(A^{-1}A)\mathbf{x} = A^{-1}\mathbf{b}$

Therefore

$$\mathbf{x} = A^{-1}b$$

$$A^{-1} = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix}$$

Therefore, the solution to the linear system in vector form is given by

$$x = A^{-1}b = \begin{bmatrix} 2 & 3 & 7 \\ -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -5 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}$$

That is $x = -1, y = 2\ and\ z = -2$