Lecture 1

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

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Subject Code: PUSASQF506A

Chapter: Unit 1 Chapter 1

Chapter Name: Introduction to life table functions

Today's Agenda

1. Lifetime Random Variables

2. Probabilities
    1. Survival Probability
    2. Mortality Probability
    3. Force of Mortality

3. The Life Table
    1. Life Table Functions

4. Life Table functions at non-integer ages
    1. Uniform Distribution of Deaths
    2. Constant Force of Mortality

5. Select Mortality

$T_x = P$

Motivation

Why do we need Survival models and Probability functions?

- We know that many insurance policies provide a benefit on the death of the policyholder.
- When an insurance company issues such a policy, the policyholder's date of death is unknown, so the insurer does not know exactly when the death benefit will be payable.
- In order to estimate the time at which a death benefit is payable, the insurer needs a model of human mortality, from which probabilities of death at particular ages can be calculated, and this is the topic of this chapter.

1 Lifetime Random Variables

Let (x) denote a life aged x, where x ≥ 0. The death of (x) can occur at any age greater than x. Future lifetime of an individual is not known to us, hence we consider it as a random variable.

We model the future lifetime of (x) by a continuous random variable which we denote by T_x.

Define
$\mathbf{T_x}$ = as the random variable for time to death for someone age x. = complete future lifetime of a life aged x

This means that x + T_x represents the age-at-death random variable for (x).

On the basis of the definition above, what does $\mathbf{T_{50} = 32.4}$ represent?

$P(T_x \le n) = F_{T_x}(n) = F_{50}(30)$

$= F_x(n)$

1 Lifetime Random Variables

Define

$F_T(t)$ = the cumulative distribution function of *T*, or P(*T* ≤ *t*).

$F_x(t)$ represents the probability that (x) does not survive beyond age *x* + *t*.

Usually, the cumulative distribution function is called the **distribution function**, dropping the word "cumulative".

Rather than using a double subscript, we will abbreviate the notation for the cumulative distribution function of T_x as $F_x(t)$.

On the basis of the above definition, what does $F_{50}(30)$ represent?

$$1 - F_x(t) = S_x(t)$$

$$1 - P(T_x \le t) = S_x(t)$$

$$P(T_x > t) = S_x(t)$$

1 Lifetime Random Variables

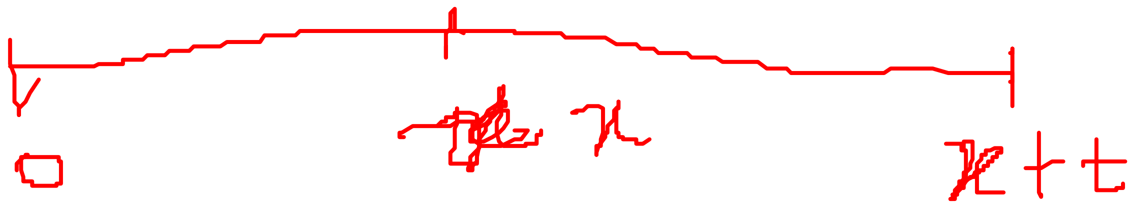
The complement of the distribution function is called the **survival function.**

Define
$S_T(t)$ = the survival distribution function of *T*, or P(*T* > *t*).

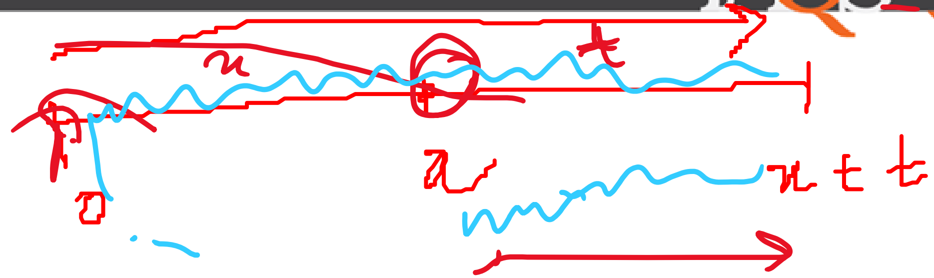
In general, $\mathbf{S_T(t) = 1 - F_T(t)}$

Again rather than using a double subscript, we will abbreviate the notation for the survival function of T_x as $S_x(t)$.

On the basis of the above definition, what does $S_{50}(30)$ represent?



1 Lifetime Random Variables



Consider T_0 and T_x for a particular individual who is now aged x. The random variable T_0 represented the future lifetime at birth for this individual, so that, at birth, the individual's age at death would have been represented by T_0. This individual could have died before reaching age x – the probability of this was $\Pr[T_0 < x]$ – but has survived.

Now that the individual has survived to age x, so that $T0 > x$, his or her future lifetime is represented by T_x and the age at death is now $x + T_x$.

If the individual dies within t years from now, then $T_x \le t$ and $T_0 \le x + t$. Loosely speaking, we require the events $[T_x \le t]$ and $[T_0 \le x + t]$ to be equivalent, given that the individual survives to age x. We achieve this by making the following assumption for all $x \ge 0$ and for all $t > 0$

$$\Pr[T_x \le t] = \Pr[T_0 \le x + t | T_0 > x].$$

$P(T_x \le t)$

$P(T_0 \le x+t)$

Thus, using the conditional probability formulae, we have

$$\Pr[T_x \le t] = \frac{P[x < T_0 \le X + t]}{P[T_0 > X]}$$

$$P(T_x \le t) = \frac{P(T_0 \le x+t) \text{ and } P(T_0 > x)}{P(T_0 > x)}$$

1 Lifetime Random Variables

$$\Pr[T_x \le t] = \frac{P[\,x < T_0 \le x + t]}{P[T_0 > X]}$$

$$= \frac{P[T_0 \le x+t] - P[T_0 \le x]}{1 - P[T_0 \le x]} \quad S_0(x)$$

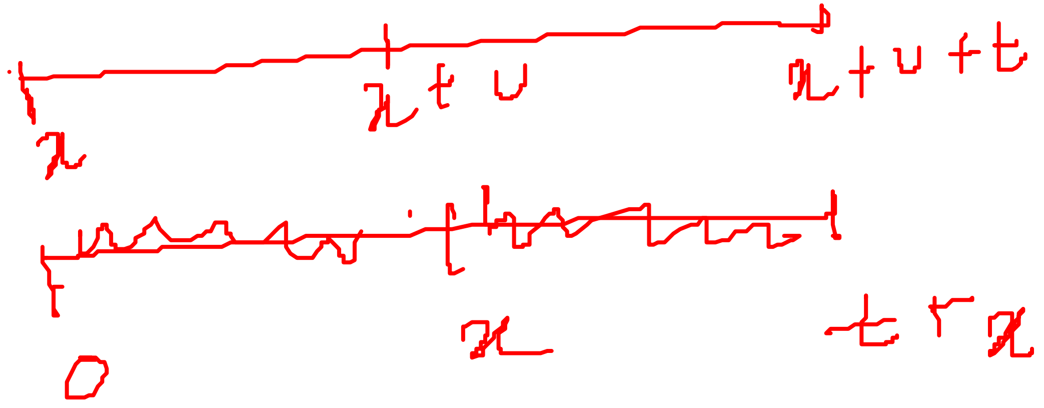
This can be written in terms of the distribution and survival function as:

$$F_x(t) = \frac{F_0(x+t) - F_0(x)}{S_0(x)}$$

Also, using $S_x(t) = 1 - F_x(t)$,

$$S_x(t) = \frac{S_0(x+t)}{S_0(x)}$$

this can be written as: $S_0(x+t) = S_0(x)S_x(t)$



This is a very important result. It shows that we can interpret the probability of survival from age x to age $x + t$ as the product of
(1) the probability of survival to age x from birth, and
(2) the probability, having survived to age x, of further surviving to age x + t

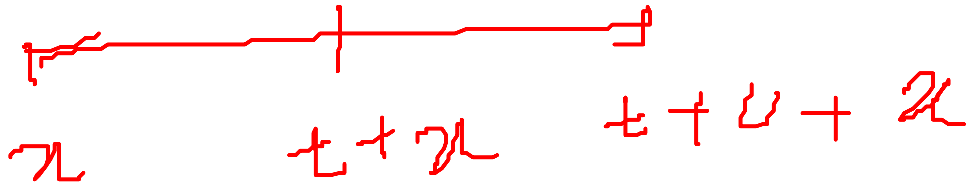
1 Lifetime Random Variables

Similarly, any survival probability for *(x)*, for, say, $t + u$ years can be split into the probability of surviving the first t years, and then, given survival to age $x + t$, subsequently surviving another u years. That is,

$$S_x(t+u) = \frac{S_0(x+t+u)}{S_0(x)}$$

$$\rightarrow\ S_x(t+u) = \frac{S_0(x+t)}{S_0(x)}\,\frac{S_0(x+t+u)}{S_0(x+t)}$$

$$\rightarrow\ S_x(t+u) = S_x(t)\,S_{x+t}(u)$$



1.1 Properties

A survival function must have the following properties:

1. $\mathbf{S_x(0) = 1.}$ Negative survival times are impossible.

2. $\mathbf{S_x(t) \geq S_x(u)}$ **for u > t.** The function is monotonically non-increasing. The probability of surviving a longer amount of time is never greater than the probability of surviving a shorter amount of time.

3. $\mathbf{\lim_{t \to \infty} S_x(t) = 0.}$ Eventually everyone dies

$x = 20 \qquad 25$

$20 \longrightarrow 20$

Question

If we wanted to express the probability that someone aged 40 will die between ages 75 and 85, how do we express this in terms of:

a. Distribution function
b. Survival function

$P(a < x < B)$

$= F(B) - F(a)$

$$P(35 < T_{40} < 45)$$

$$= F_{40}(45) - F_{40}(35)$$

$$= S_{40}(35) - S_{40}(45)$$

Solution

$$\Pr(35 < T_{40} \leq 45) = F_{40}(45) - F_{40}(35)$$

$$\Pr(35 < T_{40} \leq 45) = S_{40}(35) - S_{40}(45)$$

Question

The survival function for newborns is

$$S_0(t) = \begin{cases} \sqrt{\frac{100-t}{100}} & t \leq 100 \\ 0 & t > 100 \end{cases}$$

Calculate

1. The probability that a newborn survives to age 75 but does not survive to age 84.
2. The probability that (20) survives to age 75 but not to age 84.
3. $F_{60}(20)$.

Solution

1. We want $S_0(75) - S_0(84)$.

$$S_0(75) = \sqrt{\frac{25}{100}} = 0.5$$

$$S_0(84) = \sqrt{\frac{16}{100}} = 0.4$$

$$\Pr(75 < T_0 \leq 84) = 0.5 - 0.4 = \boxed{\mathbf{0.1}}$$

2. We want $S_{20}(55) - S_{20}(64)$.

$$S_{20}(55) = \frac{S_0(75)}{S_0(20)} = \frac{0.5}{\sqrt{80/100}} = 0.559017$$

$$S_{20}(64) = \frac{S_0(84)}{S_0(20)} = \frac{0.4}{\sqrt{80/100}} = 0.447214$$

$$\Pr(55 < T_{20} \leq 64) = 0.559017 - 0.447214 = \boxed{\mathbf{0.111803}}$$

Solution

3. $F_{60}(20) = 1 - S_{60}(20)$, and

$$S_{60}(20) = \frac{S_0(80)}{S_0(60)} = \frac{\sqrt{0.2}}{\sqrt{0.4}} = \sqrt{0.5}$$

$$F_{60}(20) = 1 - \sqrt{0.5} = \boxed{0.292893}$$

2 Probabilities

2.1 Survival Probability

It is now convenient to introduce some elementary probabilistic notions.

The first function we work with is $S_x(t) = P(T > t)$. The actuarial symbol for this is ${}_tp_x$ .The letter p denotes the concept of probability of survival. The x subscript is the age; the t pre-subscript is the duration.

$${}_tp_x = S_x(t)$$

- **${}_tp_x$ - the probability that a person aged x will survive for another t years.**

2 Probabilities

$_{u|t}q_x$

x $\quad$ $u+x$ $\quad$ $x+t+u$

2.2 Mortality Probability

The complement of the survival function is Fx(t) = P(T ≤ *t*). The actuarial symbol for this is tq_x. The letter q denotes the concept of probability of death.

$_u p_x \times {_t q_{x+u}}$

$$tq_x = F_x(t)$$

- **tq_x = the probability that person aged x will die between the ages of x and x + t.**

Deferred Mortality Probability

$$_{u|t}q_x = \Pr[u < T_x \le u + t] = S_x(u) - S_x(u + t).$$

This is the probability that *(x)* survives *u* years, and then dies in the subsequent *t* years, that is, between ages *x* + *u* and *x* + *u* + *t*. This is called a **deferred mortality probability**, because it is the probability that death occurs in some interval following a deferred period.

2.3 Force of Mortality

$A \xrightarrow{\mu_x dx+s} B$

$x \quad x+s \quad x+t \quad u+t+x+s$

The force of mortality is an important and fundamental concept in modelling future lifetime.

We denote the force of mortality at age x by μ_x and define it as the annual rate of transfer between alive and dead at exact age x , where:
$\mu_x h$ = probability of a life aged x dying over the short time interval (x, $x + h$)

ie $\mu_x h = {}_hq_x$

$T_{40} = 5$

$f_{T_x}(t) = {}_tp_x \, \mu_{x+t+s} \, ds$

More formally, we write this as:

$$\mu_x = \lim_{h\to 0} \frac{1}{h} {}_hq_x$$

$$= -\frac{1}{l_x} \lim_{h\to 0} \frac{l_{x+h} - l_x}{h}$$

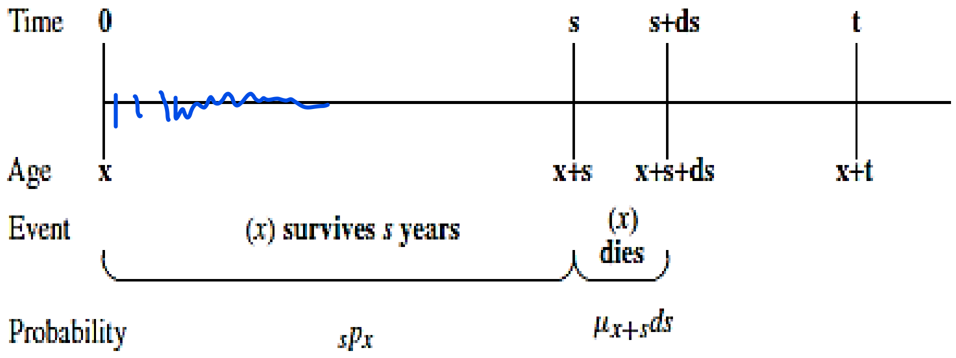
$$= -\frac{d}{d_x} \ln l_x$$

2.3 Probabilities

Important results

1. $_tp_x$ + $_tq_x$ = 1

2. $_tq_x$ = $\int_0^t {_sp_x}\, \mu_{x+s}\, ds$ = probability that a life aged dies at any of the possible moments over the next t years. This can be seen from the time line below.



3. $_tp_x$ = exp {-$\int_0^t \mu_{x+s}\, ds$}

2.4 Expected Future Lifetime

Consider the expected future lifetime of *(x)*, E[T_x], denoted in actuarial notation by $\mathring{e}_x$. We also call this the **complete expectation of life**.

E[T_x] = $\int_0^{\infty} t.\, {}_tp_x\, \mu_{x+t}\, dt$ $= \int_0^{\infty} t\, p_x\, dt$

To derive the expectation we will be use the result that,

${}_tp_x\, \mu_{x+t} = f_x(t) = \frac{d}{dt}\, {}_tq_x = \frac{d}{dt}\,(1 - {}_tp_x) = -\frac{d}{dt}\, {}_tp_x$

Therefore,

E[T_x] = $\int_0^{\infty} t.\left(-\frac{d}{dt}\, {}_tp_x\right) dt$

Using integration by parts, we have

E[T_x] = $\int_0^{\infty} t.\, {}_tp_x\, \mu_{x+t}\, dt = [t(-tpx)]_0^{\infty} - \int_{t=0}^{\infty} 1\, x\, (-tpx) dt = \int_0^{\infty} {}_tp_x\, dt = \mathring{e}_x$

$\cancel{T_x}$: $f_{T_x}(t) = {}_tp_x \mu_{x+t}\, dt \; \cancel{= f_x(t)}$

$$E(T_x) = \int_0^\infty t \cdot \cancel{{}_tp_x \mu_{x+t}\, dt}$$

$$F_x(t) = {}_tq_x = 1 - {}_tp_x$$

$$f_x(t) = 0 - \frac{d}{dt}\, {}_tp_x$$

$$E(T_x) = \int_0^\infty \underbrace{t}_{u} \cdot \underbrace{-\frac{d}{dt}\, {}_tp_x \, dt}_{v}$$

$$= -\left\{ \left[t \times {}_tp_x \right]_0^\infty - \int_0^\infty 1 \cdot {}_tp_x \, dt \right\}$$

$$= -\left\{ 0 - \int_0^\infty {}_tp_x \, dt \right\} = \int_0^\infty {}_tp_x \, dt = \mathring{e}_x$$

X x

$E(X) = \sum x p(X = x)$

2.5 Curtate Future Lifetime

The curtate future lifetime random variable is defined as the integer part of future lifetime, and is denoted by K_x for a life aged x.
Define: K_x = Int $[T_x]$

We can think of the curtate future lifetime as the number of whole years lived in the future by an individual.

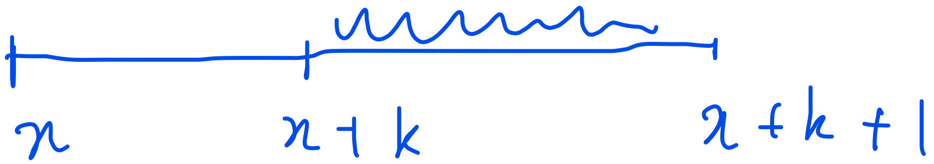
As an illustration of the importance of curtate future lifetime, consider the situation where a life aged *x* at time 0 is entitled to payments of 1 at times 1, 2, 3, . . . provided that *(x)* is alive at these times. Then the number of payments made equals the number of complete years lived after time 0 by *(x)*. This is the curtate future lifetime.

$$E(K_x) = e_x = \sum k . P(K_x = k) = \sum_{k=0}^{\infty} {}_k p_x$$

2.5 Curtate Expectation

We can find the probability function of K_x by noting that for k = 0, 1, 2, . . ., $K_x = k$ if and only if *(x)* dies between the ages of $x + k$ and $x + k + 1$.

Thus for k = 0, 1, 2, . . .



$$P[K_x = k] = P[k \le T_x < k+1]$$
$$= {}_{k|}q_x$$
$$= {}_kp_x \, q_{x+k}$$

The expected value of K_x is denoted by e_x, so that $e_x = E[K_x]$, where:

$$E[K_x] = \sum_{k=0}^{\infty} k \, P[K_x = k] = \sum_{k=0}^{\infty} k \left({}_kp_x - {}_{k+1}p_x\right) = \sum_{k=0}^{\infty} {}_kp_x = e_x$$

$$E(K_x) = \sum_{0}^{\infty} k \cdot {}_kp_x \, q_{x+k}$$

$= 0$

$+ 1 \cdot {}_1p_x \, q_{x+1}$

$+ 2 \cdot {}_2p_x \, q_{x+2}$

$+ 3 \cdot {}_3p_x \, q_{x+3}$

$\vdots$

$$= \boxed{\begin{aligned} &p_x \, q_{x+1} \\ &+ {}_2p_x \, q_{x+2} \\ &+ {}_3p_x \, q_{x+3} + \\ &+ \vdots \end{aligned}} + {}_2p_x \, q_{x+2} + {}_3p_x \, q_{x+3} + {}_3p_x \, q_{x+3}$$

$\vdots$

${}_2p_x = {}_1p_x \times p_{x+1}$

$= p_x \times 1 + {}_2p_x + {}_3p_x + \ldots$

$= \sum_{k=0}^{\infty} {}_kp_x$

$p_x \, q_{x+1} + {}_2p_x \, q_{x+2} + {}_3p_x \, q_{x+3} + \ldots$

$= p_x \left(q_{x+1} + p_{x+1} q_{x+2} + {}_2p_{x+1} q_{x+3} + \ldots \right)$

$= p_x \times 1$

$$E(K_x) = \sum_{k=0}^{\infty} k\, {}_kp_x \, q_{x+k} \times k = \sum k \left({}_kp_x - {}_{k+1}p_{x+k} \right)$$

$$= \sum 0 + 1\left({}_1p_x - {}_2p_{x+1}\right) + 2\left({}_2p_{x+1} + {}_3p_{x+2}\right) + \cdots$$

$$= p_x + 2p_x + 3p_x + \cdots$$

$$= \sum_{k=1}^{\infty} {}_kp_x$$

3 The Life Table

A **life table** (also called a **mortality table** or **actuarial table**) is a table which shows, for each age, what the probability is that a person of that age will die before their next birthday ("probability of death").
In other words, it represents the survivorship of people from a certain population.

From the starting point, a number of statistics can be derived and thus also included in the table is:

- the probability of surviving any particular year of age
- the remaining life expectancy for people at different ages etc.

For the actuary working in the life insurance field, a major objective is to estimate the mortality pattern which will be exhibited by a group of individuals. A basic device for accomplishing this is known as a life table

3 Life Table Functions

The following is an example of a portion of a life table (this is an illustration only, and the figures are not intended to be realistic):

$_2p_0 = 1$

x	ℓ_x	d_x
0	100 000	2000
1	98 000	1500
2	96 500	1000
3	95 500	900
⋮	⋮	⋮
ω	0	

The table will end at some age, traditionally denoted by ω (omega), such that $l_\omega = 0$. This is the limiting age of the table, and denotes the first age at which all of the original group will have died. The actual value of ω will vary with the particular life table, but it is typically taken to be around 110 or higher.

3.1 Life Table Functions

- l_x **- the number of lives alive at age x.**

Let l_0 be an arbitrary number, (called the radix of the table) usually taken to be a round figure such as 100 000. Suppose we start with a group of l_0 newly born lives. We would like to predict how many of these individuals will still be alive at any given time in the future.

- d_x **- the number of lives who die between age x and x+1.**

Thus, the basic relationship between the two is:

$l_{x+1} = l_x - d_x$

i.e. $d_x = l_x - l_{x+1}$

3.1 Life Table Functions

Probabilities in terms of Life table Functions

1. $_tp_x = \frac{l_{x+t}}{l_x}$

2. $_tq_x = \frac{l_x - l_{x+t}}{l_x}$ $= 1 - {}_tp_x$

3. $_{t|u}q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x}$

Questions

Table below gives an extract from a life table. Calculate

a) l_{40}
b) $_{10}p_{30}$
c) q_{35}
d) $_{5}q_{30}$
e) the probability that a life currently aged exactly 30 dies between ages 35 and 36.

x	l_x	d_x
30	10 000.00	34.78
31	9 965.22	38.10
32	9 927.12	41.76
33	9 885.35	45.81
34	9 839.55	50.26
35	9 789.29	55.17
36	9 734.12	60.56
37	9 673.56	66.49
38	9 607.07	72.99
39	9 534.08	80.11

Solution

$$l_{40} = l_{39} - d_{39} = 9\,453.97.$$

$$_{10}p_{30} = \frac{l_{40}}{l_{30}} = \frac{9\,453.97}{10\,000} = 0.94540.$$

$$q_{35} = \frac{d_{35}}{l_{35}} = \frac{55.17}{9\,789.29} = 0.00564.$$

$$_5q_{30} = \frac{l_{30} - l_{35}}{l_{30}} = 0.02107.$$

$$_{5|}q_{30} = \frac{l_{35} - l_{36}}{l_{30}} = \frac{d_{35}}{l_{30}} = 0.00552.$$

Questions

You are given:
(i) The probability that a person age 50 is alive at age 55 is 0.9.
(ii) The probability that a person age 55 is not alive at age 60 is 0.15.
(iii) The probability that a person age 50 is alive at age 65 is 0.54.

Calculate the probability that a person age 55 dies between ages 60 and 65.

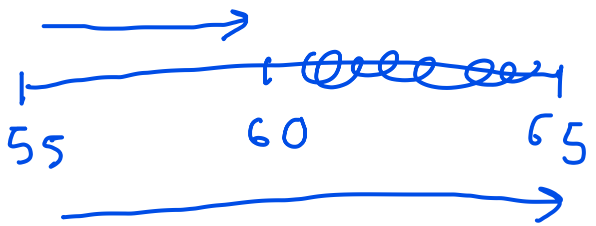
$_5 q_{50}$

$_{5|}q_{50}$

$_{u|t}q_x$

$_{5|5}q_{55}$

Solution



We need ${}_{5|5}q_{55} = {}_5p_{55} - {}_{10}p_{55}$.

$$
\begin{aligned}
{}_5p_{55} &= 1 - {}_5q_{55} = 1 - 0.15 = 0.85 \\
{}_{15}p_{50} &= 0.54 \\
{}_5p_{50}\,{}_{10}p_{55} &= 0.54 \\
0.9\,{}_{10}p_{55} &= 0.54 \\
{}_{10}p_{55} &= 0.6
\end{aligned}
$$

The answer is $0.85 - 0.6 = \boxed{\mathbf{0.25}}$.

4 Life table functions at non-integer ages

Life tables list mortality rates (q_x) or lives (l_x) for integral ages only. Often, it is necessary to determine lives at fractional ages (like $l_{x+0.5}$ for x an integer) or mortality rates for fractions of a year. We need some way to interpolate between ages.

Specifically, we need to make some assumption about the probability distribution for the future lifetime random variable between integer ages. It may be specified in terms of the force of mortality function or the survival or mortality probabilities

4.1 Uniform Distribution of Deaths

The first method is based on the assumption that, for integer x and $0 \leq t \leq 1$, the function ${}_tp_x\mu_{x+t}$ is a constant.

Since this is the density (PDF) of the time to death from age x , it is seen that this assumption is equivalent to a uniform distribution of the time to death, conditional on death falling between these two ages.

Hence it is called the Uniform Distribution of Deaths (or UDD) assumption.

4.1 Uniform Distribution of Deaths

Age is an integer & Duration is non-integer ($0 \le t \le 1$)

Under UDD, it means that the number of lives at age x+t, $0 \le t \le 1$, is a weighted average of the number of lives at age x and the number of lives at age x + 1.

$$\mathbf{l_{x+t} = (1-t)\, l_x + t.\, l_{x+1}}$$

$0.25\, q_x = 0.25 \times q_x$

Thus,

$$\begin{aligned} {}_tq_x &= 1 - {}_tp_x \\ &= 1 - \frac{l_{x+t}}{l_x} \\ &= 1 - \frac{(1-t)\, l_x + t.\, l_{x+1}}{l_x} \\ &= \frac{t\, l_x - t.\, l_{x+1}}{l_x} \\ &= t\,(1 - {}_tp_x) \\ &= \mathbf{t.\, q_x} \end{aligned}$$

4.1 Uniform Distribution of Deaths

Age & Duration both are non-integer ($0 \le s < t \le 1$)

When age and duration both are non-integer, then under UDD we have

$$_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x}$$

Derive the result

$$_{t-s}q_{x+s} = \frac{(t-s)q_x}{1-sq_x}$$

Hint: Use the principle of consistency

$$_tp_x = {}_sp_x \times {}_{t-s}p_{x+s}$$

Solution

$$ {}_{t-s}q_{x+s} = 1 - {}_{t-s}p_{x+s} = 1 - \frac{{}_tp_x}{{}_sp_x} = 1 - \frac{1 - {}_tq_x}{1 - {}_sq_x} = 1 - \frac{1 - tq_x}{1 - sq_x} $$

$$ = \frac{(1 - sq_x) - (1 - tq_x)}{1 - sq_x} = \frac{(t-s)q_x}{1 - sq_x} $$

C.F.M

$p_x = e^{-\mu}$

${}_tp_x = \exp\left[-\int_0^t \mu_{x+s}\, ds\right]$

${}_{0.3}p_{30} = \exp\left[-\int_0^{0.3} \mu_{x+0.3}\, ds\right]$

$= \exp(-0.3\mu)$

$= (p_x)^{0.3}$

Question

H · W

CT5 September 2017 Q3

Calculate $_{2.25}q_{85.5}$ using the method of Uniform Distribution of Deaths.

Basis Table: ELT15 (Males)

Solution

$$ {}_{2.25}p_{85.5} = {}_{0.5}p_{85.5}\,p_{86}\;{}_{0.75}p_{87} $$

$$ = \frac{l_{86}}{l_{85.5}} \times \frac{l_{87}}{l_{86}} \times \frac{l_{87.75}}{l_{87}} $$

$$ = \frac{l_{87.75}}{l_{85.5}} $$

$$ = \frac{0.75 l_{88} + 0.25 l_{87}}{0.5 l_{86} + 0.5 l_{85}} $$

$$ = \frac{0.75 \times 11,874 + 0.25 \times 14,280}{0.5 \times 16,917 + 0.5 \times 19,756} $$

$$ = \frac{12,475.5}{18,336.5} $$

$$ = 0.68036 $$

$$ {}_{2.25}q_{85.5} = 1 - {}_{2.25}p_{85.5} $$

$$ = 0.31964 $$

4.2 Constant Force of Mortality

The second method of approximation is based on the assumption of a *constant force of mortality*.
That is, for integer x and $0 \le t < 1$, we suppose that: $\mu_{x+t} = \mu \rightarrow$ (a constant)

Then for $0 \le s < t < 1$ the formula:

$$ {}_{t-s}p_{x+s} = \exp\{-\int_s^t \mu_{x+r}\, dr\} = e^{-(t-s)\mu} $$

can be used to find the required probabilities. We do this by first noting that, under this assumption, $p_x = e^{-\mu}$, so we can simply write:

$$ {}_{t-s}p_{x+s} = (p_x)^{(t-s)} $$

where p_x can be found from the life table.
Hence we can easily calculate any required probability.

Question

H.W

You are given:

(i) $q_x = 0.1$

(ii) The force of mortality is constant between integral ages

Calculate ${}_{1/2}q_{x+1/4}$

Solution

$$ {}_{1/2}q_{x+1/4} = 1 - {}_{1/2}p_{x+1/4} = 1 - p_x^{1/2} = 1 - 0.9^{1/2} = 1 - 0.948683 = \boxed{0.051317} $$

5 Select Mortality

Suppose you selected two 60-year-old men from the population. The first one was selected randomly, whereas the second one had recently purchased a life insurance policy. Would the mortality rate for both of these, q_{60} be the same?

5 Select Mortality

Suppose you selected two 60-year-old men from the population. The first one was selected randomly, whereas the second one had recently purchased a life insurance policy. Would the mortality rate for both of these, q_{60} be the same?

No. The second person was unde1written for a life insurance policy, which means his medical situation was reviewed. He had to satisfy certain guidelines regarding weight, blood pressure, blood lipids, family history, existing medical conditions, and possibly even driving record and credit history. The fact he was approved for an insurance policy implies that his mortality rate is lower than that of a randomly selected 60-year-old male.

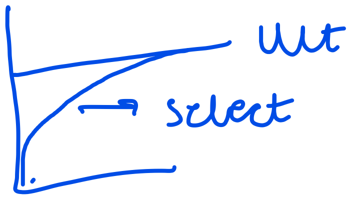
This introduces the concept of select mortality!

$q_{[40]} \longrightarrow q_{[40]+10}$

$\underline{q_{40}} \longrightarrow q_{50}$

$[40]$ ——— $[40]+10$

40 ——— 50



5 Select Mortality

A mortality table for the insured population must consider both the age of issue and the duration since issue. Mortality rates would require two arguments and need a notation like *q(x, t)* where x is the issue age and *t* the duration since issue.

International Actuarial Notation provides two-parameter notation for all actuarial functions. The parameters are written as subscripts with a bracket around the first parameter and a plus sign between the parameters. In other words, the subscript is of the form [x] + *t*. When *t* = 0, it is omitted.

Thus the mortality rate for a 60-year-old who just purchased a life insurance policy would be written as $q_{[60]}$. The mortality rate for a 65-year-old who purchased a policy at age 60 would be written $q_{[60]+5}$.

When mortality depends on the initial age as well as duration, it is known as select mortality, since the person is selected at that age.

Select 2 yrs.

$q_{[40]} \rightarrow q_{[40]+1} \rightarrow q_{42}$

$q_{40} \rightarrow q_{41} \rightarrow q_{42}$

5 Select Mortality

Working with select mortality is no different from working with non-select mortality (sometimes known as aggregate or ultimate mortality), as long as the bracketed parameter is not changed.

Question

You are given:
$l_{[45]} = 1000$
$_5q_{[45]} = 0.04$
$_5q_{[45]+5} = 0.05$

Calculate $l_{[45]+10}$ $= l_{[45]} \times (1 - {_5q_{[45]}}) \times (1 - {_5q_{[45]+5}})$

$=$

Solution

$$l_{[45]+10} = l_{[45]}\,{}_5p_{[45]}\,{}_5p_{[45]+5} = (1000)(0.96)(0.95) = \boxed{912}$$

5 Select Mortality

5.1 Reading the Table

A select-and-ultimate mortality table is shown in tabular form by listing the issue ages vertically and the durations horizontally. If the select period is n years, there are n columns for n durations, followed by a column with ultimate mortality. The columns are arranged so that to find the mortality at all durations for a specific issue age, you read across the row corresponding to that issue age and when you hit the end of the row, you continue from that cell down the last column

Notice that there is no direct way to go from $[x]$ to $[x + l]$. A person who is selected at age x will be $[x]$ in the first year, $[x] + 1$ in the second year, and so on, until he becomes ultimate at age $x + k$, and then his age will continue to increase and be without brackets. He will never be selected again. He will never be $[x + l]$.

Question

'H · W

You are given:
(i) Mortality. rates are select and ultimate with a select period of 3 years, and are given in Table.
(ii) $l_{[40]}$ =10,000,000.

Compute $l_{[42]}$

x	$q_{[x]}$	$q_{[x]+1}$	$q_{[x]+2}$	q_{x+3}	$x+3$
40	0.002	0.005	0.008	0.012	43
41	0.003	0.006	0.009	0.015	44
42	0.004	0.007	0.010	0.018	45

Solution

We must compute l_{45} recursively.

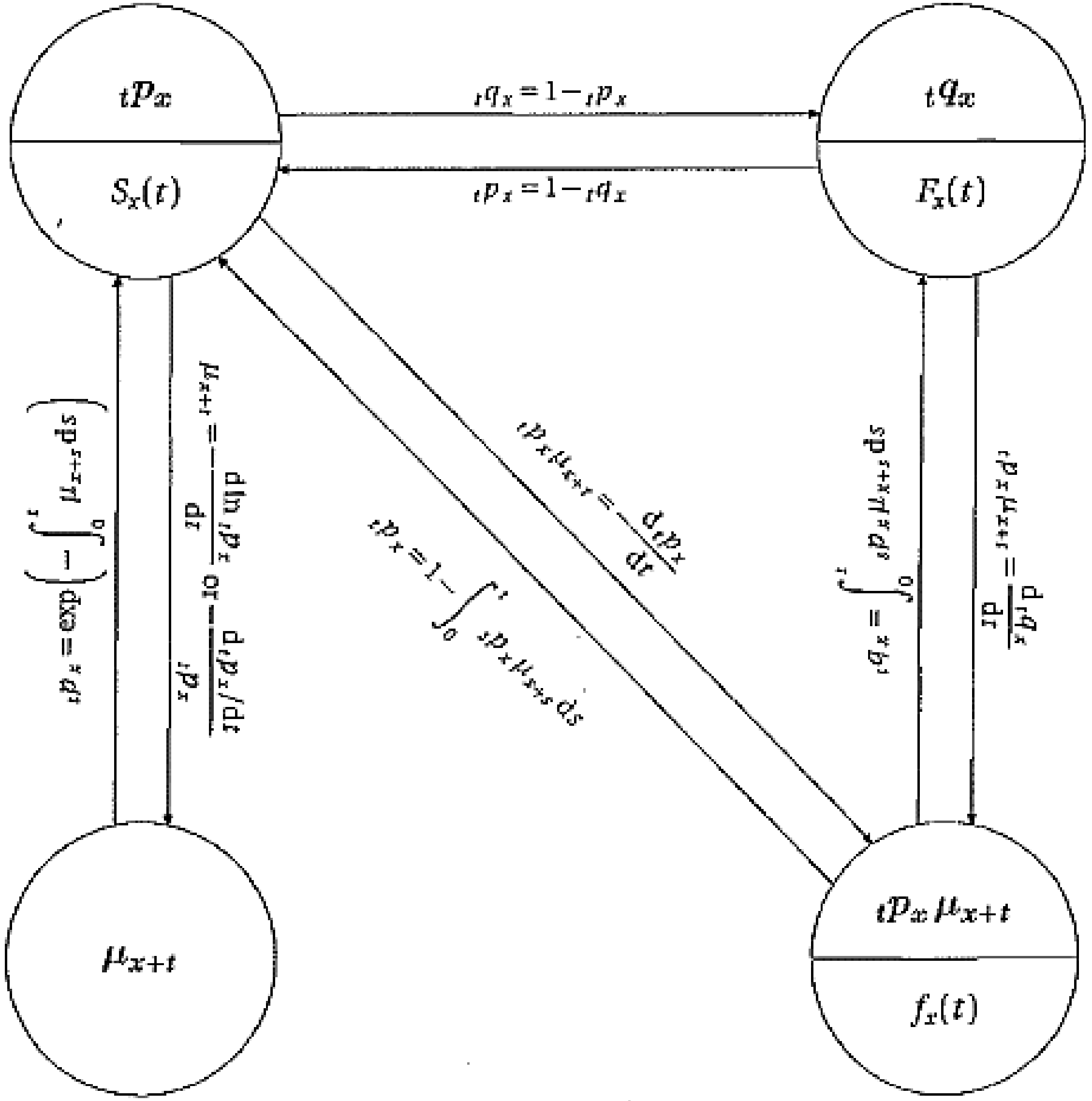
$$l_{[40]+1} = 10{,}000{,}000(1-0.002) = 9{,}980{,}000$$
$$l_{[40]+2} = 9{,}980{,}000(1-0.005) = 9{,}930{,}100$$
$$l_{43} = 9{,}930{,}100(1-0.008) = 9{,}850{,}659$$
$$l_{44} = 9{,}850{,}659(1-0.012) = 9{,}732{,}451$$
$$l_{45} = 9{,}732{,}451(1-0.015) = 9{,}586{,}464$$

Now we work backwards from l_{45} to $l_{[42]}$.

$$l_{[42]+2} = \frac{9{,}586{,}464}{1-0.010} = 9{,}683{,}297$$
$$l_{[42]+1} = \frac{9{,}683{,}297}{1-0.007} = 9{,}751{,}558$$
$$l_{[42]} = \frac{9{,}751{,}558}{1-0.004} = \boxed{9{,}790{,}721}$$

Summary

The following diagram summarizes the survival model:



Summary

Important results:

1. Complete Expectation of life
$E[T_x] = \int_0^\infty {}_tp_x \, dt = \mathring{e}_x$

2. Curtate Expectation of Life
$E[K_x] = \sum_{k=0}^{\infty} {}_kp_x = e_x$

3. $\mathring{e}_x = e_x + 1/2$

For Fractional ages

1. Under UDD assumption
(a) ${}_tq_x = t.q_x$
(b) ${}_{t-s}q_{x+s} = \frac{(t-s)q_x}{1 - sq_x}$

2. Under CFM assumption
${}_{t-s}p_{x+s} = (p_x)^{(t-s)}$

Question - Homework

H.W

CT5 September 2005 Q4

Calculate the value of $_{1.75}p_{45.5}$ on the basis of mortality of AM92 Ultimate and assuming that deaths are uniformly distributed between integral ages.

CT5 September 2005 Q5

A population is subject to a constant force of mortality of 0.015.
Calculate:
(a) The probability that a life aged 20 exact will die before age 21.25 exact.

CT5 September 2015 Q1

Calculate:
(a) $_{25}p_{40}$
(b) $_{10|}q_{[53]}$

Question

The table below is part of a mortality table used by a life insurance company to calculate probabilities for a special type of life insurance policy.

x	$l_{[x]}$	$l_{[x]+1}$	$l_{[x]+2}$	$l_{[x]+3}$	l_{x+4}
51	1,537	1,517	1,502	1,492	1,483
52	1,532	1,512	1,497	1,487	1,477
53	1,525	1,505	1,490	1,480	1,470
54	1,517	1,499	1,484	1,474	1,462
55	1,512	1,492	1,477	1,467	1,453

$${}_{t|u}q_x = \frac{l_{x+t} - l_{x+t+u}}{l_x}$$

(i) Calculate the probability that a policyholder who was accepted for insurance exactly 2 years ago and is now aged exactly 55 will die between age 56 and age 57. ${}_{1|1}q_{[53]+2}$

(ii) Calculate the corresponding probability for an individual of the same age who has been a policyholder for many years. ${}_{1|1}q_{55}$

(iii) Comment on your answers to (i) and (ii).

Solution

(i) *The table in the question is not laid out in the same way as AM92 in the Tables.*

The policyholder is currently aged $[53]+2$. So the probability of dying between ages 56 and 57 is:

$$\frac{l_{[53]+3}-l_{57}}{l_{[53]+2}}=\frac{1,480-1,470}{1,490}=0.00671$$

(ii) The corresponding probability for an ultimate policyholder is:

$$\frac{l_{56}-l_{57}}{l_{55}}=\frac{1,477-1,470}{1,483}=0.00472$$

(iii) For the usual types of policies (life assurance and annuities), policyholders experience lighter mortality during the select period. Here, however, the mortality rate is higher in (i) than in (ii), so these policyholders experience heavier mortality during the select period. This could occur, for example, if the policy was sold to individuals who had recently had a particular form of medical treatment that increased mortality rates during the first few years.