INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc - Semester 3

Subject : Pricing and Reserving for Life Insurance Products

Chapter: Unit 1 Chapter 2

Chapter Name: Life Assurance and Annuity Contracts

Today’s Agenda

0. Introduction

1. Life Assurance Contracts
    1. Whole Life Assurance
    2. Term Assurance
    3. Pure Endowment
    4. Endowment Assurance Contracts
    5. Deferred Assurance

1. Life Annuity Contracts
    1. Whole Life Annuities
    2. Temporary Annuities
    3. Deferred Annuities
    4. Guaranteed Annuities
    5. Continuous Annuities

1. Premium Conversion Formulae

0 Introduction

Discuss:
What are assurance contracts? What are annuity contracts? And what is the difference between the two benefits?

0 Introduction

- **A life insurance pays a benefit (single) to the insured policyholder upon death or survival.**
- The benefit may be payable whenever the insured dies, or may be payable only if the insured dies within a fixed number of years, or may be payable only if the insured dies after a fixed number of years. The benefit may vary depending on when death occurs.

- **A life annuity contract provides payments of amounts, which might be level or variable, at stated times, provided a life is still then alive.**

0 Motivation

- We are interested in calculating the expected value of the present value of the benefit. This is the amount we would put aside today to fund the benefit.

- For a large group of independent policyholders, if we set aside the expected present value of the benefit for each policyholder, the law of large numbers tells us that we will have approximately enough money to pay all the benefits when they become due.

- We would also like to calculate the variance of the present value of the benefit. Then we can use the normal approximation to determine the size of the fund that has a high probability of being adequate to pay all the benefits.

Revision

1. Curtate future lifetime random variable, $\boldsymbol{K_x}$
It measures the number of complete years of future life of *(x)*.
The probability function for K_x is given as, P[K_x = k] = k|q_x

2. Continuous future lifetime random variable, $\boldsymbol{T_x}$
It measures the number of exact years of future life of *(x)*.
The probability density function for T_x is given as, f_x(t) = tp_x. μ_{x+t}

Note: (x) represents some life aged x now.

0 Introduction

Types of Insurance Contracts
We will consider valuation of the following kind of contracts:

Whole life assurance

- The benefit under such a contract is an amount, called the sum assured, which will be paid on the policyholder's death.

Term assurance

- A term assurance contract is a contract to pay a sum assured on or after death, provided death occurs during a specified period, called the term of the contract.

Pure Endowment

- A pure endowment contract provides a sum assured at the end of a fixed term, provided the policyholder is alive.

Endowment assurance

- A sum assured is payable either on death during the term or on survival to the end of the term. The sums assured payable on death or survival need not be the same, although they often are.

1.1 Whole Life Assurance

The benefit under such a contract is an amount, called the sum assured, which will be paid on the policyholder's death.

1.1.A Expected Present Value

Suppose that the benefit of $1 is payable at the end of the year of death of (x), To value this we use the curtate future lifetime random variable, K_x , it measures the number of complete years of future life of (x).
The time to the end of the year of death of (x) is then K_x +1.

We use Z to denote the present value of the insurance benefit, so that Z is the random variable.
Present Value Random Variable: $Z = v^{K_x+1}$

The **EPV of the benefit**, E[Z], is denoted by A_x in actuarial notation.

$$A_x = E[Z] = E[v^{K_x+1}] = \sum_{k=0}^{\infty} v^{k+1}\ {}_{k|}q_x = v\, q_x + v^2\ {}_{1|}q_x + v^3\ {}_{2|}q_x + \ldots..$$

1.1 Whole Life Assurance

When the sum assured is S, then we have

Z = $S\, v^{K_x+1}$

The EPV of the benefit is,

$$E[Z] = E[S.\, v^{K_x+1}] = S.E[\, v^{K_x+1}] = S.A_x$$

Generalization

In General, we can always express the EPV of a life-contingent benefit by considering each time point at which the benefit can be paid, and summing over all possible payment times the product of:

(1) the amount of the benefit,
(2) the appropriate discount factor, and
(3) the probability that the benefit will be paid at that time.

1.1 Whole Life Assurance

1.1.B Variance of Present Value Random Variable

A useful method for calculating higher moments of standard insurances is that the k"' moment of a standard insurance equals the first moment calculated at k times the force of interest. This works whenever the death benefit in all years is either 0 or 1. This method is called the rule of moments.

The Variance of $Z = v^{K_x+1}$ can be given as:

$$Var[Z] = var[v^{K_x+1}] = E[(v^{K_x+1})^2] - [E(v^{K_x+1})]^2$$

1.1 Whole Life Assurance

The Variance of Z = v^{K_x+1} can be given as:

$\text{Var}[Z] = \text{var}[v^{K_x+1}] = E[(v^{K_x+1})^2] - [E\left(v^{K_x+1}\right)]^2$

Since $(v^{K_x+1})^2 = (v^2)^{K_x+1}$ the first term is just ${}^2A_x$ where the '2' prefix denotes an EPV calculated at a rate of interest $(1+i)^2$-1, or at a force of interest 2δ.

So, $\text{var}[v^{K_x+1}] = {}^2A_x - (A_x)^2$

The variance of the present value of a benefit of S payable at the end of the year of death is:

$\text{var}[Sv^{K_x+1}] = S^2\,\text{var}[v^{K_x+1}] = S^2[{}^2A_x - (A_x)^2]$

Question

An insurance on [70] pays 1000 at the end of the year of death if [70] dies after 1 year but not after 3 years.

You are given:

(i) Mortality is based on the following 2-year select-and-ultimate table:

x	$q_{[x]}$	$q_{[x]+1}$	q_{x+2}	$x+2$
70	0.05	0.07	0.10	72
71	0.06	0.08	0.12	73

(ii) i = 0.04

(iii) Z is the present value random variable for the insurance.

Calculate E[Z].

Solution

$$\mathrm{E}[Z] = 1000\left(\frac{{}_{1|}q_{[70]}}{1.04^2} + \frac{{}_{2|}q_{[70]}}{1.04^3}\right)$$

We have

$$ {}_{1|}q_{[70]} = p_{[70]}\, q_{[70]+1} = (0.95)(0.07) = 0.0665$$
$$ {}_{2|}q_{[70]} = p_{[70]}\, p_{[70]+1}\, q_{72} = (0.95)(0.93)(0.10) = 0.08835$$

So the answer is

$$\mathrm{E}[Z] = 1000\left(\frac{0.0665}{1.04^2} + \frac{0.08835}{1.04^3}\right) = 1000(0.061483 + 0.078543) = \boxed{\mathbf{140.03}}.$$

Question

Assume the same mortality and interest as the previous question. An insurance on [70] pays a benefit of 1000 at the end of the year if death occurs in the first year and 2000 at the end of the year if death occurs in the second year. Let Z be the payment random variable for this insurance.

Calculate Var(Z).

Solution

Answer: The first moment is

$$E[Z] = 1000\left(\frac{0.05}{1.04}\right) + 2000\left(\frac{0.0665}{1.04^2}\right) = 48.07692 + 122.96598 = 171.04290$$

When calculating the second moment, the benefit as well as the interest factor must be squared.

$$E[Z^2] = 1000^2\left(\frac{0.05}{1.04^2}\right) + 2000^2\left(\frac{0.0665}{1.04^4}\right) = 46{,}227.81 + 227{,}377.91 = 273{,}605.72$$

$$\mathrm{Var}(Z) = 273{,}605.72 - 171.04290^2 = \boxed{244{,}350.05}$$

1.1.C Benefits payable immediately on death

So far we have assumed that assurance death benefits have been paid at the end of the year of death. In practice, assurance death benefits are paid a short time after death, as soon as the validity of the claim can be verified.

Thus, Real life insurance pays the benefit soon after the death of the policyholder.

Assuming a delay until the end of the year of death is therefore not a prudent approximation, but assuming that there is no delay and that the sum assured is paid immediately on death is a prudent approximation.

Try to write the present value random variable for whole life assurance, assuming benefit of $1 is payable immediately on death.

1.1.C Benefits payable immediately on death

For life *(x)*, the **present value** of a benefit of $1 payable immediately on death is a random variable, *Z*, say, where

$Z = v^{T_x} = e^{-\delta T_x}$

The **EPV of the benefit**, E[*Z*], is denoted by $\bar{A}_x$ in actuarial notation.

$$\bar{A}_x = E[Z] = E[e^{-\delta T_x}] = \int_0^\infty e^{-\delta t}\, {}_tp_x .\, \mu_{x+t}\, dt$$

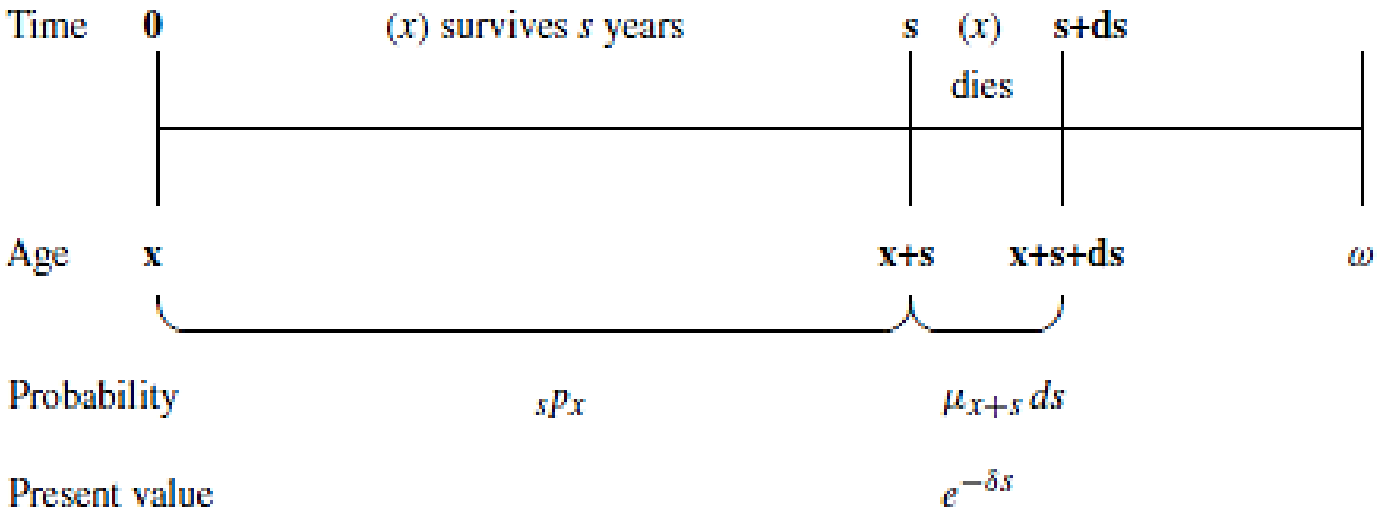
Variance of the benefit

$${}^2\bar{A}_x = \int_0^\infty e^{-2\delta t}\, {}_tp_x .\, \mu_{x+t}\, dt$$

$$Var[Z] = E[Z^2] - E[Z]^2 = {}^2\bar{A}_x - (\bar{A}_x)^2$$

Discuss

Given below is the timeline for whole life assurance with benefit payable immediately on death. With the help of it, discuss the intuition/logic behind the EPV of the benefit.



Discuss

Here, we are assuming that the interest rate is fixed and Constant. Do you think that is appropriate?

1.2 Term Assurance

Under a term insurance policy, the death benefit is payable only if the policyholder dies within a fixed term of, say, n years.

1.2.A Expected Present Value

We consider the situation when a death benefit of 1 is payable at the end of the year of death, provided this occurs within n years.

The **Present Value Random Variable** is : Z = $\begin{cases} v^{K_x+1} & \text{if } K_x \le n-1 \\ 0 & \text{if } K_x > n \end{cases}$

The EPV of the benefit, E[Z], is denoted by $A^{1}_{x:\overline{n|}}$ in actuarial notation.

$$A^{1}_{x:\overline{n|}} = E[Z] = E[v^{K_x+1}] = \sum_{k=0}^{n-1} v^{k+1} \; {}_{k|}q_x$$

1.2 Term Assurance

The EPV of the benefit is denoted by $A^1_{x:\overline{n}|}$ in actuarial notation.

The notation has two statuses: x and n

The exact condition for payment is identified by the number that is placed above the statuses (*ie* the 1) and where it is placed:

- the number is positioned above the life status x : this indicates that the payment is made *only* when the life status x fails (*ie* dies)
- the number over the x is 1: this tells us that the life status x has to fail *first* out of the two statuses involved, in order for the payment to be made.

1.2 Term Assurance

1.2.B Variance of Present Value Random Variable

Along the same lines as for the whole life assurance:

The Variance of Z can be given as:

Var[Z] = ${}^{2}A^{1}_{x:\overline{n|}}$ - $(A^{1}_{x:\overline{n|}})^2$

where the '2' prefix denotes an EPV calculated at rate of interest $(1+i)^2$-1.

1.2.C Benefits payable immediately on death

For life *(x)*, the **present value** of a benefit of \$1 payable immediately on death is a random variable, *Z*, say, where

$$Z = \begin{cases} v^{T_x} = e^{-\delta T_x} & \text{if } T_x \le n \\ 0 & \text{if } T_x > n \end{cases}$$

The **EPV of the benefit**, E[*Z*], is denoted by $\bar{A}^{1}_{x:\overline{n|}}$ in actuarial notation.

$$\bar{A}^{1}_{x:\overline{n|}} = \mathrm{E}[Z] = \mathrm{E}[e^{-\delta T_x}] = \int_0^n e^{-\delta t}\, {}_tp_x \cdot \mu_{x+t}\, dt$$

Variance of the benefit

$${}^2\bar{A}^{1}_{x:\overline{n|}} = \int_0^n e^{-2\delta t}\, {}_tp_x \cdot \mu_{x+t}\, dt$$

$$\mathrm{Var}[Z] = \mathrm{E}[Z^2] - \mathrm{E}[Z]^2 = {}^2\bar{A}^{1}_{x:\overline{n|}} - (\bar{A}^{1}_{x:\overline{n|}})^2$$

Question

CT5 September 2017
A special whole life assurance policy is issued on a life aged 50 exact.

Under this policy the sum assured, payable at the end of the year of death, is 1 unit for the first 10 years decreasing to 0.75 units thereafter.

(i) Calculate the expected present value of the benefit.

(ii) Determine the variance of the present value of the benefit.

Basis:
Mortality AM92 Ultimate
Interest 4% per annum

Solution

$$\text{EPV} = A^{1}_{50:\overline{10|}} + 0.75 \times v^{10} \times \frac{l_{60}}{l_{50}} \times A_{60}$$

$$= (A_{50} - v^{10} \times \frac{l_{60}}{l_{50}} \times A_{60}) + 0.75 \times v^{10} \times \frac{l_{60}}{l_{50}} \times A_{60}$$

$$= A_{50} - 0.25 \times v^{10} \times \frac{l_{60}}{l_{50}} \times A_{60}$$

$$= 0.32907 - 0.25 \times 0.67556 \times \frac{9287.2164}{9712.0728} \times 0.45640$$

$$= 0.25536$$

Solution

First calculate 2nd moment

$$\text{Value} = ({}^2A_{50} - v^{20} \times \frac{l_{60}}{l_{50}} \times {}^2A_{60}) + (0.75)^2 \times v^{20} \times \frac{l_{60}}{l_{50}} \times {}^2A_{60}$$

$$= {}^2A_{50} - 0.4375 \times v^{20} \times \frac{l_{60}}{l_{50}} \times {}^2A_{60}$$

$$= 0.13065 - 0.4375 \times 0.45639 \times \frac{9287.2164}{9712.0728} \times 0.23723$$

$$= 0.08535$$

$$\text{Variance} = 0.08535 - (0.25536)^2 = 0.02014 = (0.14192)^2$$

Question

Using the Standard Ultimate Survival Model, with interest at 5% per year effective, calculate $A^{1}_{x:\,\overline{10|}}$ for x = 20, 40, 60 and 80 and comment on the values.

Solution

x	$A_{x:\overline{10}\rceil}^{\,1}$
20	0.00209
40	0.00573
60	0.04252
80	0.33722

We observe that values increase as *x* increases, reflecting the fact that the probability of death in a 10- year period increases with age for the survival model we are using.

1.3 Pure Endowment

A pure endowment contract provides a sum assured at the end of a fixed term, provided the policyholder is then alive.

1.3.A Expected Present Value

We consider the situation when a benefit of 1 is payable at the end of the term, provided the policyholder is alive then.

The **Present Value Random Variable** is : Z = $\begin{cases} 0 & \text{if } K_x < n \\ v^n & \text{if } K_x \geq n \end{cases}$

The EPV of the benefit, E[Z], is denoted by $A_{x:\overset{\;1}{\overline{n|}}}$ in actuarial notation.

$$A_{x:\overset{\;1}{\overline{n|}}} = E[Z] = 0.\,P[K_x < n] + v^n.P[K_x \geq n]$$
$$= v^n.{}_np_x$$

1.3 Pure Endowment

The EPV of the benefit is denoted by $A_{x:\overset{\,1}{\overline{n|}}}$ in actuarial notation.

The notation has two statuses: x and n

- the benefit is paid only when the term of n years ends, ie at the moment at which the n status fails, so the number (whatever it may be) needs to be placed above the n
- the benefit will only be paid (at time n) if the person is still alive at that time: this requires the status n to be the first of the two statuses to fail, and hence we need the number to be a ‘1’.

1.3 Pure Endowment

1.3.B Variance of Present Value Random Variable

Along the same lines:

The Variance of Z can be given as:

$$\text{Var}[Z] = {}^{2}A_{x:\overset{\;1}{\overline{n|}}} - (A_{x:\overset{\;1}{\overline{n|}}})^2$$

where the '2' prefix denotes an EPV calculated at rate of interest $(1+i)^2$-1.

1.4 Endowment Assurance Contracts

An endowment assurance provides a combination of a term assurance and a pure endowment.

The sum insured is payable on the death of (x) should (x) die within a fixed term, say n years, but if (x) survives for n years, the sum insured is payable at the end of the nth year.

Consider the case when the death benefit (of amount 1) is payable at the end of year od death, if (x) dies or at the end of the n year term, if (x) survives.

The **Present Value Random Variable Z**, is given as: = $\begin{cases} v^{K_x+1} & \text{if } K_x \leq \text{n-1} \\ v^n & \text{if } K_x \ \text{n} \end{cases}$

$$= v^{\min(K_x+1,n)}$$

1.4 Endowment Assurance Contracts

1.4.A Expected Present Value

The EPV of the benefit, E[*Z*], is denoted by $A_{x:\overline{n}|}$ in actuarial notation.

The EPV of the benefit is then:

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x + v^n.P[K_x \geq n]$$

$$= \sum_{k=0}^{n-1} v^{k+1}\ {}_{k|}q_x + v^n.{}_np_x$$

$$= A^1_{x:\overline{n}|} + A_{x:\overset{1}{\overline{n}|}}$$

1.4 Endowment Assurance Contracts

1.4.B Variance of Present Value Random Variable

Term assurance and Pure Endowment are not independent random variables (one must be zero and the other non-zero).
This is because the life will either survive to the end of the *n*-year period or die during it.

Therefore we must calculate Variance from first principles.

Along the same lines:

The Variance of Z can be given as:
Var[Z] = ${}^{2}A_{x:\overline{n|}}$ - $(A_{x:\overline{n|}})^2$

where the '2' prefix denotes an EPV calculated at rate of interest $(1+i)^2$-1.

1.4.C Benefit Payable Immediately on death

Consider the case when the death benefit (of amount $1) is payable immediately on death.
The **present value** of the benefit is Z, say, where

$$Z = \begin{cases} v^{T_x} & \text{if } T_x < n \\ v^n & \text{if } T_x \geq n \end{cases}$$

$$= v^{\min(T_x,n)} = e^{-\delta \min(T_x,n)}$$

Thus, the **EPV of the benefit** is

$$E[Z] = \int_0^n e^{-\delta t}\, {}_tp_x .\, \mu_{x+t}\, dt + e^{-\delta n}\, {}_np_x$$

$$\bar{A}_{x:\,\overline{n|}} = \bar{A}^{1}_{x:\,\overline{n|}} + A_{x:\,\overset{\;1}{\overline{n|}}}$$

It is only death benefits that are affected by the changed time of payment. Survival benefits such as a pure endowment are not affected.

1.4.C Benefit Payable Immediately on death

Variance of the Present Value

The expected value of the squared present value of the benefit is

$$ {}^{2}\bar{A}_{x:\overline{n}|} = \int_0^n e^{-2\delta t}\, {}_tp_x \cdot \mu_{x+t}\, dt + e^{-2\delta n}\, {}_np_x $$

Therefore:

$$ \text{Var}[Z] = {}^{2}\bar{A}_{x:\overline{n}|} - (\bar{A}_{x:\overline{n}|})^2 $$

Question

Using the Standard Ultimate Survival Model, with interest at 5% per year effective, calculate $A^{1}_{x:\,\overline{10}|}$ for x = 20, 40, 60 and 80 and comment on the values.

Solution

x	$A_{x:\overline{10}\rvert}$
20	0.61433
40	0.61494
60	0.62116
80	0.67674

The actuarial values of the 10-year endowment insurance functions do not vary greatly with x, unlike the values of the 10-year term insurance functions.

The reason for this is that the probability of surviving 10 years is large ($_{10}p_{20}$ = 0.9973, $_{10}p_{60}$ = 0.9425) and so for each value of x, the benefit is payable after 10 years with a high probability

1.5 Deferred Assurance Contracts

Deferred insurance refers to insurance which does not begin to offer death benefit cover until the end of a deferred period

1.5.A Deferred Whole Life Assurance

A whole life assurance with sum assured 1, payable to a life aged *x* but deferred *n* years is a contract to pay a death benefit of 1 provided death occurs after age $x + n$.

If we let *Z* denote the **present value** of this benefit, then: $Z = \begin{cases} 0 & \text{if } K_x < n \\ v^{K_x+1} & \text{if } K_x \geq n \end{cases}$

1.5 Deferred Assurance Contracts

1.5.A Deferred Whole Life Assurance

Expected Present Value

If the benefit is payable at the end of the year of death (if at all), the EPV of this assurance is denoted $_{n|}A_x$.

As usual, the subscript of $n|$ to the left of the symbol indicates that the event is deferred for n years.

$$_{n|}A_x = A_x - A^{1}_{x:\overline{n|}} = v^n . {}_np_x A_{x+n}$$

The factor $v^n . {}_np_x$ is important and useful in developing EPVs. It plays the role of the pure interest discount factor v^n, where now the payment or present value being discounted depends on the survival of a life aged x.

For benefit payable immediately on death:

$$_{n|}\bar{A}_x = \bar{A}_x - \bar{A}^{1}_{x:\overline{n|}} = v^n . {}_np_x \bar{A}_{x+n}$$

1.5 Deferred Assurance Contracts

Deferred Whole Life Assurance

Variance of the present value random variable

Let *X* be the present value of a whole life assurance and *Y* the present value of a term assurance with term *n* years, both for a sum assured of 1 payable at the end of the year of death of a life aged *x* . Then
E[X] = A_x and E[Y] = $A^1_{x:\overline{n|}}$

Thus E[X-Y] = A_x - $A^1_{x:\overline{n|}}$ and

Var[X –Y] = var[X] + var [Y] - 2cov[X,Y]

where cov[X,Y] = ${}^2A^1_{x:\overline{n|}}$ - $A_x\, A^1_{x:\overline{n|}}$

1.5 Deferred Assurance Contracts

1.5.A Deferred Whole Life Assurance

Variance of the present value random variable

Var[X –Y] = var[X] + var [Y] - 2cov[X,Y]

$$= {}^2A_x - (A_x)^2 + {}^2A^1_{x:\overline{n|}} - (A^1_{x:\overline{n|}})^2 - 2({}^2A^1_{x:\overline{n|}} - A_x\, A^1_{x:\overline{n|}})$$

$$= {}^2A_x - (n|A_x)^2 - {}^2A^1_{x:\overline{n|}}$$

$$= n|{}^2A_x - (n|A_x)^2$$

1.5 Deferred Assurance Contracts

1.5.B Deferred Term Assurance

Suppose a benefit of $1 is payable at the end of the year of the death of *(x)* provided that *(x)* dies between ages $x + m$ and $x + m + n$.

The **present value random variable** is

$$Z = \begin{cases} 0 & \text{if } K_x < m \text{ or } K_x \geq m+n \\ v^{K_x+1} & \text{if } m \leq K_x < m+n \end{cases}$$

The **Expected Present Value** denoted by, $m|A^1_{x:\overline{n}|}$ is given as:

$$m|A^1_{x:\overline{n}|} = A^1_{x:\overline{m+n}|} - A^1_{x:\overline{n}|} = v^m . {}_mp_x \, A^1_{x+m:\overline{n}|}$$

1.5 Deferred Assurance Contracts

1.5.B Deferred Term Assurance

Variance of the present value random variable

This is given by the formula

$$\mathrm{Var}[Z] = {}_{m|}{}^{2}A^{1}_{x:\overline{n|}} - ({}_{m|}A^{1}_{x:\overline{n|}})^{2}$$

1.6 Claims Acceleration Process

It is convenient to be able to estimate $\bar{A}_x$, $\bar{A}^{1}_{x:\overline{n|}}$ and so on in terms of commonly tabulated functions. One simple approximation is ***claims acceleration***.

Of deaths occurring between ages $x + k$ and $x + k + 1$, say, ($k = 0, 1, 2, \ldots$) roughly speaking the average age at death will be $x + k + ½$. Under this assumption **claims are paid on average 6 months before the end of the year of death.**

E.g. $\bar{A}_x \approx v^{1/2}\, q_x + v^{1\,1/2}\, {}_{1|}q_x + v^{2\,1/2}\, {}_{2|}q_x + \ldots..$

$= (1+i)^{1/2}\, (v\, q_x + v^2\, {}_{1|}q_x + v^3\, {}_{2|}q_x + \ldots..)$

$= (1+i)^{1/2}\, A_x$

1.6 Claims Acceleration Process

We obtain the approximate EPVs:

$$\bar{A}_x = (1+i)^{1/2}\, A_x$$

$$\bar{A}^{1}_{x:\overline{n|}} = (1+i)^{1/2}\, A^{1}_{x:\overline{n|}}$$

$$\bar{A}_{x:\overline{n|}} = (1+i)^{1/2}\, A^{1}_{x:\overline{n|}} + A_{x:\overset{1}{\overline{n|}}}$$

1.6 Further Approximation

A second approximation is obtained by considering a whole life or term assurance to be a sum of deferred term assurances, each for a term of one year. Then, taking the whole life case as an example:

$$\bar{A}_x = \bar{A}^1_{x:\,\overline{1|}} + \text{v px}\ \bar{A}^1_{x+1:\,\overline{1|}} + v^2\ \text{2px}\ \bar{A}^1_{x+2:\,\overline{1|}} + \ldots\ldots$$

Now, $\bar{A}^1_{x+1:\,\overline{1|}} = \int_0^1 v^t\ tp_{x+k}\, \mu_{x+k+t}\, dt$

If we now make the assumption that deaths are uniformly distributed between integer ages, then

$q_{x+k} = \int_0^1 tp_{x+k}\, \mu_{x+k+t}\, dt = tp_{x+k}\, \mu_{x+k+t}$ (Since under UDD, $tp_{x+k}\, \mu_{x+k+t}$ is a constant)

Thus, $\bar{A}^1_{x+k:\,\overline{1|}} \approx q_{x+k} \int_0^1 v^t\ dt = q_{x+k} \frac{i.v}{\delta}$

Hence:

$\bar{A}_x = \frac{i}{\delta}$ (v $q_x + v^2\, p_x\, q_{x+1} + \ldots.$)

$$\boldsymbol{\bar{A}_x = \frac{i}{\delta} A_x}$$

1.7 Functions for Select Lives

- We have developed results in terms of lives subject to ultimate mortality. We have taken this approach simply for ease of presentation.
- All of the above development equally applies to lives subject to select mortality.
- For example, $\bar{A}_{[x]}$ denotes the EPV of a benefit of 1 payable immediately on the death of a select life $[x]$.
- Similarly, $A^{1}_{[x]:\overline{n|}}$ denotes the EPV of a benefit of 1 payable at the end of the year of death of a select life $[x]$ should death occur within n years.

1.8 Evaluating Assurance Benefits

We saw earlier that

$_{n|}A_x = A_x - A^{1}_{x:\overline{n|}} = v^n . {}_np_x A_{x+n}$

Rearranging gives us:

$A^{1}_{x:\overline{n|}} = A_x - v^n . {}_np_x A_{x+n}$

The function A_x is tabulated in all the tables and hence calculation can be easy.

Question

Calculate $\bar{A}_{47:\overline{11|}}$

Basis: Mortality AM92
Interest 4% per annum

Solution

$$\bar{A}_{47:\overline{11}|} = (1.04)^{1/2} \times A^{1}_{47:\overline{11}|} + (1.04)^{-11} \times \frac{l_{58}}{l_{47}}$$

$$= 1.0198 \times \left(A_{47} - (1.04)^{-11} \times \frac{l_{58}}{l_{47}} \times A_{58}\right) + (1.04)^{-11} \times \frac{l_{58}}{l_{47}}$$

$$= 1.0198 \times \left(0.29635 - 0.64958 \times \frac{9413.8004}{9771.0789} \times 0.42896\right) + 0.64958 \times \frac{9413.8004}{9771.0789}$$

$$= 0.02845 + 0.62583$$

$$= 0.65428$$

2 Life Annuity Contracts

The term **life annuity** to refer to a series of payments to (or from) an individual as long as the individual is alive on the payment date.

The valuation of annuities is important as annuities appear in the calculation of premiums, policy values and pension benefits.

2 Life Annuity Contracts

Types of Annuity Contracts

We consider four varieties of life annuity contract:

Whole life level annuity	Temporary level annuity	Deferred annuity.	Guaranteed annuity.
• Annuities under which payments are made for the whole of life, with level payments, called a whole life level annuity or, more usually, an immediate annuity.	• Annuities under which level payments are made only during a limited term, called a temporary level annuity or, more usually, just a temporary annuity.	• Annuities under which the start of payment is deferred for a given term, called a deferred annuity.	• Annuities under which payments are made for the whole of life, or for a given term if longer, called a guaranteed annuity.

Further, we consider the possibilities that payments are made in advance or in arrears.

2.1 Whole life annuities payable annually in arrears

An immediate annuity is one under which the first payment is made within the first year.

Consider an annuity contract to pay 1 at the end of each future year, provided a life now aged x is then alive.

Present Value Random Variable

If the life dies between ages $x + k$ and $x + k + 1$ ($k = 0,\ldots, \omega - x - 1$) which is to say, $K_x = k$, the present value at time 0 of the annuity payments which are made is $a_{\overline{k}|}$. (We define $a_{\overline{0}|} = 0$.)
Therefore the **present value at time 0 of the annuity payments is $a_{\overline{K_x}|}$**.

2.1 Whole life annuities payable annually in arrears

2.1.A Expected present value

The expected value of $a_{\overline{K_x|}}$ is:

$$\mathbf{E}[a_{\overline{K_x|}}] = \sum_{k=0}^{\infty} a_{\overline{k|}} \, P[K_x = k]$$

Actuarial notation for the expected present value

a_x = EPV of an immediate annuity of 1 unit pa paid in arrears for as long as (x) remains alive in the future. The word 'expected' really means that we're making an allowance for the probability of payment.

$$a_x = \mathbf{E}[a_{\overline{K_x|}}] = \sum_{k=0}^{\infty} a_{\overline{k|}} \; {}_{k|}q_x$$

We can write this in a form that is easier to calculate.

2.1 Whole life annuities payable annually in arrears

We write:

$$a_x = \sum_{k=0}^{\infty}\left(\sum_{k=j+1}^{\infty} k|q_x\right)v^{j+1}$$

Now, in general

$$\sum_{k=j+1}^{\infty} k|q_x = {}_{j+1}p_x$$

Hence,

$$a_x = \sum_{j=0}^{\infty} {}_{j+1}p_x \, v^{j+1} = \sum_{j=1}^{\infty} {}_{j}p_x \, v^{j}$$

Each payment is conditional on whether the policyholder is alive or not at the time the payment is due. The present value of the annuity payment made at time j is v^j. The expected present value is the product of the two.

The timeline below explains the annuity payments.

Time	0	1	2	3	...
Amount		1	1	1	
Discount		v	v^2	v^3	
Probability		p_x	${}_2p_x$	${}_3p_x$	

2.1 Whole life annuities payable annually in arrears

2.1.B Variance of Present Value Random Variable

We know that,

$$a_{\overline{n|}} = \frac{1 - v^n}{i}$$

Therefore:

$$\text{Var}[a_{\overline{K_x|}}] = \text{Var}\,[\frac{1 - v^{Kx}}{i}] = \frac{1}{i^2}\,\text{Var}[v^{kx}]$$

Multiply and divide by v

$$\text{Var}[a_{\overline{K_x|}}] = \frac{1}{i^2}\,\text{Var}[\frac{v^{kx+1}}{v}]$$

$$= \frac{1}{i^2\,v^2}\,\text{Var}\,[v^{Kx+1}]$$

$$= \frac{1}{d^2}\,[{}^2A_x - (A_x)^2]$$

2.1 Whole life annuities payable annually in advance

An annuity-due is one under which payments are made in advance.

Consider an annuity contract to pay 1 at the start of each future year, provided a life now aged x is then alive.

Present Value Random Variable
The **present value at time 0 of the annuity payments is** $\ddot{a}_{\overline{K_x+1}|}$.

For the annuity in advance, there will be one additional payment at the start of the contract, so there are $Kx + 1$ payments altogether.

2.1 Whole life annuities payable annually in advance

2.1.C Expected Present Value

In actuarial notation we denote the EPV,

$$\mathrm{E}[\ddot{a}_{\overline{K_x+1|}}] = \ddot{a}_x$$

We have:

$$\ddot{a}_x = \sum_{j=0}^{\infty} {}_jp_x \; v^j$$

2.1 Whole life annuities payable annually in arrears

2.1.D Variance of Present Value Random Variable

$$\text{Var}[\ddot{a}_{\overline{K_x+1}|}] = \text{Var}\left[\frac{1 - v^{Kx+1}}{d}\right]$$

$$= \frac{1}{d^2}\,\text{Var}\,[v^{Kx+1}]$$

$$= \frac{1}{d^2}\,[{}^2A_x - (A_x)^2]$$

2.1 Whole life annuities - Relationship

What is the relation between a_x and $\ddot{a}_x$?

$$\ddot{a}_x = a_x + 1$$

Intuitively, the annuity-due is the same as the annuity payable in arrears, except for the additional payment of 1 unit made at outset. This payment has present value 1, and will definitely be paid, so its expected present value must also be 1.

2.2 Temporary annuities payable annually in arrears

A temporary immediate annuity differs from a whole life immediate annuity in that the payments are limited to a specified term.

Consider a temporary annuity contract to pay 1 at the end of each future year, for the next n years, provided a life now aged x is then alive.

Present Value Random Variable

If X denotes the present value, therefore X = $\begin{cases} a_{\overline{K_x}|} & \text{if } K_x < n \\ a_{\overline{n}|} & \text{if } K_x \geq n \end{cases}$

Therefore the **present value at time 0 of the benefit is $a_{\overline{min(K_x,n)}|}$**.

2.2 Temporary annuities payable annually in arrears

2.2.A Expected Present Value

In actuarial notation we denote the EPV,

$$E[a_{\overline{min(K_x,n)}|}] = a_{x:\overline{n}|}$$

We have:

$$a_{x:\overline{n}|} = \sum_{j=1}^{n} {}_jp_x \, v^j$$

The logic behind the formula is same as that for whole life annuity. $a_{x:\overline{n}|}$ is the same summation but allows only for the first *n* payments

2.2 Temporary annuities payable annually in arrears

2.2.B Variance of Present Value Random Variable

$$\text{Var}[a_{\overline{min(K_x,n)}|}] = \frac{1}{d^2}[{}^2A^{1}_{x:\overline{n+1}|} - (A^{1}_{x:\overline{n+1}|})^2]$$

2.2 Temporary annuities payable annually in advance

A temporary immediate annuity-due has payments that are made in advance and are limited to a specified term.

Consider a temporary immediate annuity-due contract to pay 1 at the start of each of the next *n* years, provided a life now aged *x* is then alive.

Present Value Random Variable

If X denotes the present value, therefore X = $\begin{cases} \ddot{a}_{\overline{K_x+1|}} & \text{if } K_x < n \\ \ddot{a}_{\overline{n|}} & \text{if } K_x \geq n \end{cases}$

Therefore the **present value at time 0 of the benefit is** $\ddot{a}_{\overline{min(K_x+1,n)|}}$.

2.2 Temporary annuities payable annually in advance

2.2.C Expected Present Value

In actuarial notation we denote the EPV,

$$\mathrm{E}[\ddot{a}_{\overline{min(K_x+1,n)}|}] = \ddot{a}_{x:\overline{n}|}$$

We have:

$$\ddot{a}_{x:\overline{n}|} = \sum_{j=0}^{n-1} {}_{j}p_x \, v^j$$

The logic behind the formula is same as that for whole life annuity. $\ddot{a}_{x:\overline{n}|}$ is the same summation but with payments continuing only up to time $n-1$ (making n payments in total).

2.2 Temporary annuities payable annually in advance

2.2.D Variance of Present Value Random Variable

$$\text{Var}[\ddot{a}_{\overline{min(K_x+1,n)}|}] = \frac{1}{d^2}[{}^2A^{1}_{x:\overline{n}|} - (A^{1}_{x:\overline{n}|})^2]$$

2.2 Temporary annuities - Relationship

What is the relation between $\ddot{a}_{x:\overline{n|}}$ and $a_{x:\overline{n|}}$?

$$\ddot{a}_{x:\overline{n|}} - a_{x:\overline{n|}} = 1 - v^n {}_np_x$$

By expanding the summations for both, try to prove the above relationship.

2.3 Deferred Annuities - Arrears

Deferred annuities are annuities under which payment does not begin immediately but is deferred for one or more years.

Consider, an annuity of 1 per annum payable annually in arrears to a life now aged x , deferred for n years. Payment will be at ages $x + n + 1$, $x + n + 2$, .. , provided that the life survives to these ages.

The **Present Value Random Variable**, say Z, is:

$$Z = \begin{cases} 0 & \text{if } K_x \leq n \\ v^n \, a_{\overline{Kx - n|}} & \text{if } K_x > n \end{cases}$$

$$= v^n a_{\overline{\max(Kx - n, 0)|}}$$

2.3 Deferred Annuities - Arrears

2.3.A Expected Present Value

To find, the expected present value of the annuity benefit, we use the same logic as that for deferred assurances.
A n-year deferred annuity is equal to a whole life annuity less a n-year temporary annuity.

In actuarial notation, the EPV of this deferred annuity is denoted as n|a_x, where

$$_{n|}a_x = E[Z] = a_x - a_{x:\overline{n|}}$$

This can be written in another way as:

$$_{n|}a_x = v^n \, {}_np_x \, a_{x+n}$$

2.3 Deferred Annuities - Due

Consider, an annuity of 1 per annum payable annually in advance to a life now aged x , deferred for n years. Payment will be at ages $x + n$, $x + n + 1$, $x + n + 2$, .. , provided that the life survives to these ages.

Similar to arrears, the expected present value can be given as:

$$\mathbf{n|}\ddot{a}_x = \ddot{a}_x - \ddot{a}_{x:\overline{n|}} = v^n \, {}_np_x \, \ddot{a}_{x+n}$$

Note that

$$a_x = {}_{1|}\ddot{a}_x$$

2.3 Deferred Annuities - Due

2.3.B Variance of the present value random variable

The variance of this benefit is:

$$\mathrm{Var}[\ddot{a}_{\overline{\max(Kx+1,n)}|}] = \mathrm{var}[\ \frac{1-v^{\max(Kx+1,n)}}{d}\ ]$$

$$=\frac{1}{d^2}\{\ \mathrm{E}[(v^{\max(Kx+1,n)})^2] - (E\left[v^{\max(Kx+1,n)}\right])^2\ \}$$

$$=\frac{1}{d^2}(\ v^{2n} nq_x\ n|^2A_x - (v^n nq_x\ n|A_x)^2\ )$$

2.4 Guaranteed Annuities - Due

A guaranteed annuity differs from a whole life annuity in that the payments have a minimum specified term.

Consider a guaranteed annuity contract to pay 1 at the start of each future year for the next *n* years, and at the start of each subsequent future year provided a life now aged *x* is then alive.

The **present value of this benefit** is

$$Z = \begin{cases} \ddot{a}_{\overline{n|}} & \text{if } K_x < n \\ \ddot{a}_{\overline{K_x+1|}} & \text{if } K_x \geq n \end{cases}$$

$$= \ddot{a}_{\overline{\max(K_x+1,n)|}}$$

2.4 Guaranteed Annuities - Due

2.4.A Expected Present Value

The timeline for a guaranteed annual annuity-due is given as:

Time	0	1	2	……	n-1	n	n+1	…
Amount	1	1	1		1	1	1	
Discount	1	v^1	v^2		v^{n-1}	v^n	v^{n+1}	
Probability	1	1	1		1	${}_{n}p_x$	${}_{n+1}p_x$	

In actuarial notation, the EPV of this guaranteed annuity is denoted as $\ddot{a}_{\overline{\overline{x:\overline{n}|}}}$ where

$$\ddot{a}_{\overline{\overline{x:\overline{n}|}}} = E[\ddot{a}_{\overline{\max(K_x+1,n)}|}]$$

2.4 Guaranteed Annuities - Due

2.4.A Expected Present Value

$$\ddot{a}_{\overline{x:\,\overline{n|}}} = \mathrm{E}[\ddot{a}_{\overline{\max(Kx+1,n)|}}] = \sum_{k=0}^{n-1} \ddot{a}_{\overline{n|}}\ k|q_x + \sum_{k=n}^{\infty} \ddot{a}_{\overline{k+1|}}\ k|q_x$$

$$= \ddot{a}_{\overline{n|}} + \mathrm{n}|\ddot{a}_x$$

$$= \ddot{a}_{\overline{n|}} + v^n\ \mathrm{n}p_x\ \ddot{a}_{x+n}$$

2.4 Guaranteed Annuities - Due

2.4.B Notation

$\ddot{a}_{\overline{x:\overline{n|}}}$

The combined status $\overline{u:v}$ (*ie* with a bar) means a status that is active while *either or both* of the individual statuses *u* and *v* are active. It is known as the *last survivor* status,

When applied to an annuity, $\ddot{a}_{\overline{x:\overline{n|}}}$, it implies that payments continue until the last surviving status fails. Payments continue until the *later* of the death of (*x*) , or the expiry of *n* years.

2.4 Guaranteed Annuities - Due

2.4.C Variance of the present value random variable

The variance of this benefit is:

$$\text{Var}[\ddot{a}_{\overline{\max(Kx+1,n)}|}] = \text{var}[\ \frac{1 - v^{\max(Kx+1,n)}}{d}\ ]$$

$$= \frac{1}{d^2}\ \{\ \text{E}[(v^{\max(Kx+1,n)})^2] - (E[v^{\max(Kx+1,n)}])^2\ \}$$

$$= \frac{1}{d^2}\ (\ v^{2n} n q_x\ n|^2 A_x - (v^n n q_x\ n|A_x)^2\ )$$

Discuss

Variance for the deferred annuity-due and the variance of the guaranteed annuity-due are the same. Discuss why?

2.4 Guaranteed Annuities - Arrears

Consider a guaranteed annuity contract to pay 1 at the end of each future year for the next *n* years, and at the end of each subsequent future year provided a life now aged *x* is then alive.

The present value of this benefit is $a_{\overline{\max(Kx,\, n)}|}$.

2.4.C Expected Present Value

The expected value of E[$a_{\overline{\max(Kx,\, n)}|}$**] is given as**

$$\boldsymbol{a_{\overline{\overline{x:\, n}|}} = a_{\overline{n}|} + n|a_x}$$

2.4 Guaranteed Annuities - Arrears

2.4.D Variance of the present value random variable

The variance of this benefit is:

$$\text{Var}[a_{\overline{\max(Kx,\, n)}|}] = \text{Var}[\ddot{a}_{\overline{\max(Kx+1,n+1)}|} \; - 1]$$

$$= \text{var}[\,\frac{1 - v^{\max(Kx+1,n+1)}}{d}\,]$$

$$= \frac{1}{d^2}\,(\, v^{2(n+1)}(n+1)q_x\;(n+1)|^2A_x \text{ - } (v^{(n+1)}(n+1)q_x\;(n+1)|A_x)^2\,)$$

2.5 Continuous Annuities

In practice annuities are payable at discrete time intervals, but if these intervals are close together, for example weekly, it is convenient to treat payments as being made continuously.

Immediate annuity

Consider an immediate annuity of 1 per annum payable continuously during the lifetime of a life now aged x .

Present value random variable

The present value of this annuity is $\bar{a}_{\overline{T_x}|}$.

2.5 Continuous Annuities

Expected Present value

The EPV of the benefit, denoted by $\bar{a}_x$, is:

$$\bar{a}_x = \int_0^\infty e^{-\delta t}\, {}_tp_x \; dt$$

The approach is to use the sum (here an integral) of the product of the amount paid in each infinitesimal interval *(t, t +dt)*, the discount factor for the interval and the probability that the payment is made.

2.5 Continuous Annuities

Expected Present value

Another approach to find EPV is

$$\bar{a}_x = \frac{1 - E[v^{T_x}]}{\delta}$$

$$\bar{a}_x = \frac{1 - \bar{A}_x}{\delta}$$

Variance of Present Value Random Variable

$$\text{Var}[\bar{\boldsymbol{a}}_{\overline{\boldsymbol{T_x}|}}] = \frac{1}{\delta^2}\text{Var}[v^{T_x}]$$

$$= \frac{1}{\delta^2}\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$$

Note: $\bar{a}_{x:\overline{n|}} = \bar{a}_x - v^n \, {}_np_x \, \bar{a}_{x+n}$

2.6 Approximations

To evaluate continuous annuities, use the approximation:

$\bar{a}_x \approx \ddot{a}_x - ½$

$\bar{a}_x \approx a_x + ½$

Question

Calculate:

$\bar{a}_{55:\overline{10|}}$

Basis:
Mortality AM92
Interest 4% per annum

Solution

$$\bar{a}_{55:\overline{10|}} = (\ddot{a}_{55} - 0.5 - v^{10}(l_{65} / l_{55})(\ddot{a}_{65} - 0.5))$$

$$= 15.873 - 0.5 - 0.67556 \times \left(\frac{8821.2612}{9557.8179}\right) \times (12.276 - 0.5)$$

$$= 8.031$$

2.7 Evaluating Annuity Benefits

Relationships between annuity functions

$\ddot{a}_x = 1 + a_x$

$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|}$

$\ddot{a}_{x:\overline{n}|} = \ddot{a}_x - v^n\, {}_np_x\, \ddot{a}_{x+n}$ (similar for annuity arrears)

$a_x = v\, p_x\, \ddot{a}_{x+1}$

$a_{x:\overline{n}|} = v\, p_x\, \ddot{a}_{x+1:\overline{n}|}$

$\bar{a}_{x:\overline{n}|} = \bar{a}_x - v^n\, {}_np_x\, \bar{a}_{x+n}$

$a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n\, {}_np_x$

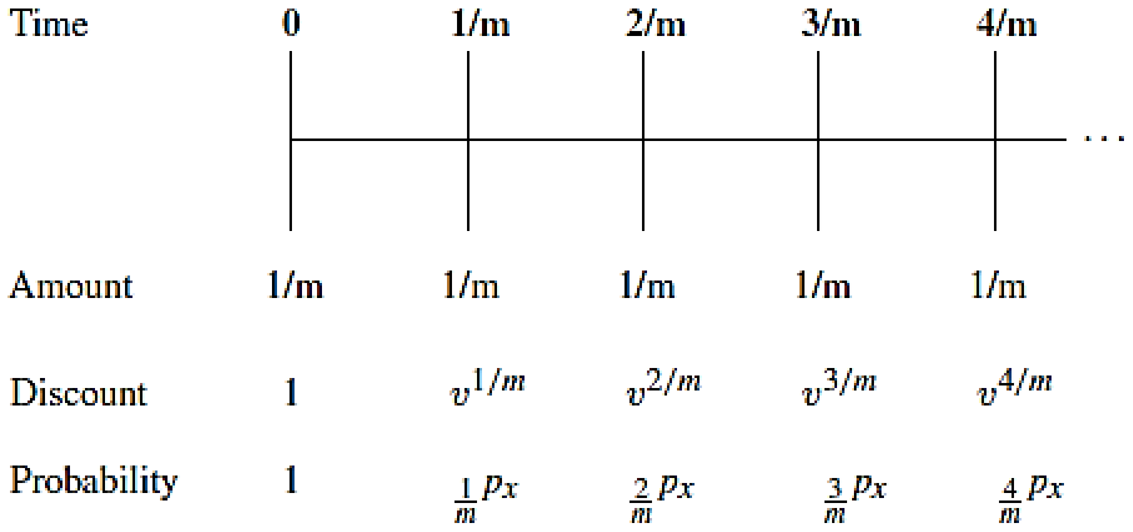
2.8 Annuities Payable m times per year

For premiums, annuities and pension benefits, the annual form of the annuity would be unusual.

We now consider the question of how annuities, with payments made more than once each year but less frequently than continuously, may be evaluated.

2.8.A Whole life mthly annuity due
Consider first an annuity of total amount 1 per year, payable in advance *m* times per year throughout the lifetime of *(x)*, with each payment being 1/*m*.

This is plotted on the time line below:



2.8 Annuities Payable m times per year

Whole life mthly annuity due
Consider first an annuity of total amount 1 per year, payable in advance *m* times per year throughout the lifetime of *(x)*, with each payment being 1/*m*.

The EPV is denoted as $\ddot{a}_x^{(m)}$, given as:

$$\ddot{a}_x^{(m)} = \sum_{t=0}^{\infty} \frac{1}{m} v^{t/m} \, {}_{t/m}p_x$$

More often, an approximation will be needed to evaluate the expression.

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{(m-1)}{2m}$$

For annuities payable 1/*m*thly in arrear, we can use the following relationship: (since the only difference in the is the first payment, of 1/*m*)

$$a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m}$$

2.8 Annuities Payable m times per year

2.8.B Temporary annuities payable mthly

Similar to whole life, approximations for temporary annuities can be developed:

$$\ddot{a}^{(m)}_{x:\overline{n|}} = \ddot{a}^{(m)}_{x} - v^n \, {}_{n}p_x \, \ddot{a}^{(m)}_{x+n}$$

Relation between arrears and due

$$a^{(m)}_{x:\overline{n|}} = \ddot{a}^{(m)}_{x:\overline{n|}} - \frac{1}{m}\left(1 - v^n \, {}_{n}p_x\right)$$

3 Premium Conversion Formulae

3.1 Discrete Case

$\ddot{a}_x$ = E[$\ddot{a}_{\overline{Kx+1}|}$] = $\frac{1-E[v^{kx+1}]}{d}$ = $\frac{1-A_x}{d}$

Thus, A_x = 1 - d$\ddot{a}_x$

Similarly,
$A_{x:\overline{n}|}$ = 1 - d$\ddot{a}_{x:\overline{n}|}$

The above relations hold for select lives also.

3.2 Continuous Case

$\bar{a}_x$ = E[$\bar{a}_{\overline{Tx}|}$] = $\frac{1-E[v^{Tx}]}{\delta}$ = $\frac{1-\bar{A}_x}{\delta}$

Thus, $\bar{A}_x$ = 1 - δ$\bar{a}_x$

Similarly,
$\bar{A}_{x:\overline{n}|}$ = 1 - δ$\bar{a}_{x:\overline{n}|}$

The above relations hold for select lives also.

Question

CT5 September 2015

A special annuity pays 5,000 per annum for five years increasing to 6,000 per annum for the next five years and increasing further to 7,000 thereafter. The payments for the first five years are guaranteed and thereafter are contingent on survival. The annuity is payable monthly in advance.

Calculate the expected present value of this annuity for a life aged 60 exact. Show all your workings.

Basis:
Mortality PMA92C20
Interest 4% per annum

Solution

$$\text{EPV} = 5000\ddot{a}^{(12)}_{\overline{5}|} + 6000\,{}_{5|}\ddot{a}^{(12)}_{60} + 1000\,{}_{10|}\ddot{a}^{(12)}_{60}$$

$$= 5000 \times \left(\frac{1-v^5}{d^{(12)}}\right) + 6000 \times v^5\,{}_5p_{60}\left(\ddot{a}_{65} - 11/24\right) + 1000 \times v^{10}\,{}_{10}p_{60}\left(\ddot{a}_{70} - 11/24\right)$$

$$= 5000 \times \left(\frac{1-v^5}{0.039157}\right) + 6000v^5 \times \frac{9647.797}{9826.131} \times \left(13.666 - \frac{11}{24}\right)$$

$$+1000v^{10} \times \frac{9238.134}{9826.131} \times \left(11.562 - \frac{11}{24}\right)$$

$$= 22738.32 + 63952.31 + 7052.36$$

$$= 93743 \text{ rounded}$$

Question

CT5 April 2010

100 graduates aged 21 exact decide to place the sum of £1 per week into a fund to be shared on their retirement at age 66 exact.

Show that each surviving member can expect to receive on retirement a fund of approximately £7,240.

Basis:
Rate of interest 4% per annum
Mortality AM92 Ultimate

Solution

$$\text{Fund} = 52 * \frac{1.04^{(66-21)}\,\bar{a}_{21:\overline{45}|}}{{}_{45}p_{21}}$$

$$\bar{a}_{21:\overline{45}|} = \ddot{a}_{21:\overline{45}|} - \frac{1}{2} * (1 - v^{45} * l_{66} / l_{21}) = \ddot{a}_{21:\overline{45}|} - \frac{1}{2} * \left(1 - 0.17120 * \frac{8695.6199}{9976.3909}\right)$$

$$= \ddot{a}_{21:\overline{45}|} - 0.42539$$

$$\ddot{a}_{21:\overline{45}|} = \ddot{a}_{21:\overline{44}|} + v^{44} * l_{65} / l_{21} = 21.045 + .17805 * \frac{8821.2612}{9976.3909} = 21.202$$

$$\Rightarrow \bar{a}_{21:\overline{45}|} = 20.777$$

$$\text{therefore fund } = \frac{52 * 1.04^{45}(20.777)}{\left(8695.6199 / 9976.3909\right)} = 7,240$$

Question

CT5 April 2015

Calculate $A_{50:\overline{4}|}$

Basis:

Mortality $q_{50} = 0.05$

$q_{51} = 0.06$

$q_{51+t} = 1.1 q_{50+t}$ for $t \geq 1$

Interest 6% per annum

Solution

$$\ddot{a}_{50:\overline{4}|} = 1+\frac{(1-.05)}{1.06}+\frac{(1-.05)(1-.06)}{(1.06)^2}+\frac{(1-.05)(1-.06)(1-.06(1.1))}{(1.06)^3}$$

$$= 1+0.89623+0.79477+0.70029=3.39129$$

$$A_{50:\overline{4}|} = 1-d(6\%)\ddot{a}_{50:\overline{4}|} = 1-\frac{.06}{1.06}(3.39129)=0.80804$$

Question

CT5 September 2017

(i) Calculate $\ddot{a}_{40:\overline{4|}}$

(ii) Derive the value of $A^{1}_{40:\overline{4|}}$, using your result from part (i).

Interest 5% per annum

Basis:
From the following life table extract

x	l_x
40	100,000
41	99,200
42	98,100
43	96,700
44	94,700

Solution

(i)

$$\ddot{a}_{40:\overline{4}|} = 1 + \frac{0.992}{1.05} + \frac{0.981}{(1.05)^2} + \frac{0.967}{(1.05)^3}$$

$$= 1 + 0.94476 + 0.88980 + 0.83533$$

$$= 3.6699$$

(ii)

$$A_{40:\overline{4}|} = 1 - d(5\%) \times \ddot{a}_{40:\overline{4}|} = 1 - \frac{0.05}{1.05} \times 3.6699$$

$$= 0.82524$$

$$A^{1}_{40:\overline{4}|} = A_{40:\overline{4}|} - (1.05)^{-4} \times \frac{l_{44}}{l_{40}} = 0.82524 - 0.82270 \times 0.947 = 0.04614$$