Lecture 3

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: TY BSc

Subject : Pricing and Reserving for Life Insurance Products

Chapter: Unit 2

Chapter Name: Net Premiums and Reserving

Today’s Agenda

0. Introduction

1. Future Loss Random Variable

1. Principle of Equivalence

1. Calculating premiums that satisfy probabilities

1. Reserves
    1. Why hold reserves
    2. Prospective Reserves
    3. Reserve Conventions
    4. Notations
    5. Retrospective Reserves

2. Equality of Prospective and Retrospective Reserves

1. Recursive relationship between reserves

Today’s Agenda

8. Mortality Profit
    1. Death Strain at Risk
    2. Expected DSAR
    3. Actual DSAR
    4. Mortality profit
    5. Mortality profit on a portfolio of policies
    6. Allowing for survival benefits
    7. Annuities

0 Introduction

An insurance policy is a financial agreement between the insurance company and the policyholder. The **insurance company agrees to pay some benefits**, for example a sum insured on the death of the policyholder within the term of a term insurance, and the **policyholder agrees to pay premiums** to the insurance company to secure these benefits. The premiums will also need to reimburse the insurance company for the expenses associated with the policy.

If the calculation does explicitly allow for expenses, the premium is called a **gross premium** or **office premium**.

If the calculation does not explicitly allow for expenses, the premium is called a **net premium or risk premium.**

0 Introduction

The premium may be a single payment by the policyholder – a **single premium** – or it may be a regular series of payments, possibly annually, quarterly, monthly or weekly.

It is common for **regular premiums to be a level amount**, but they do not have to be.

The premium paying term for a policy is the maximum length of time for which premiums are payable. The premium paying term may be the same as the term of the policy, but it could be shorter.

A key feature of any life insurance policy is that **premiums are payable in advance**, with the first premium payable when the policy is purchased.

?

Why do you think premiums are payable in advance? Why not in arrears?

0 Introduction

A key feature of any life insurance policy is that premiums are payable in advance, with the first premium payable when the policy is purchased.

To see why this is necessary, suppose it were possible to purchase a whole life insurance policy with annual premiums where the first premium were payable at the end of the year in which the policy was purchased. In this case, a person could purchase the policy and then withdraw from the contract at the end of the first year before paying the premium then due. This person would have had a year of insurance cover without paying anything for it.

Hence premiums are paid in advance.

1 Future Loss Random Variable

Consider the net random future loss (or just 'net loss') from a policy which is in force – where the net loss, L , is defined to be:
L = present value of the future outgo – present value of the future income

L is a random variable, since both terms are random variables which depend on the policyholder's future lifetime. (If premiums are not being paid, the second term is zero but the first term is a random variable, so L is still a random variable.)

1 Future Loss Random Variable - Example

Whole life insurance

An insurer issues a whole life insurance to 60, with sum insured S payable immediately on death. Premiums are payable annually in advance, ceasing at age 80 or on earlier death. The net annual premium is P.

Write down the net future loss random variable, L, for this contract in terms of lifetime random variables for 60.

1 Future Loss Random Variable - Example

Whole life insurance

An insurer issues a whole life insurance to 60, with sum insured *S* payable immediately on death. Premiums are payable annually in advance, ceasing at age 80 or on earlier death. The net annual premium is *P*.

Write down the net future loss random variable, *L*, for this contract in terms of lifetime random variables for 60.

We know that the present value random variable for the benefit is $Sv^{T_{60}}$ and that the present value random variable for the premium income is $P\ddot{a}_{\overline{\min[K_{60}+1,20]|}}$.

Thus,

$$L = Sv^{T_{60}} - P\ddot{a}_{\overline{\min[K_{60}+1,20]|}}$$

1 Future Loss Random Variable - Example

Endowment insurance

An insurer issues a 25-year endowment insurance policy to life aged 45, with sum insured *S* payable immediately on death.
Level premiums are payable continuously. Expenses to be ignored. The net annual premium is *P*.

Write down the net future loss random variable, *L*, for this contract in terms of lifetime random variables for 45.

1 Future Loss Random Variable - Example

Endowment insurance

An insurer issues a 25-year endowment insurance policy to life aged 45, with sum insured *S* payable immediately on death.
Level premiums are payable continuously. Expenses to be ignored. The net annual premium is *P*.
Write down the net future loss random variable, *L*, for this contract in terms of lifetime random variables for 45.

In a case like this it is usually easiest to write down the FLRV in two parts: the first being the outcome if the person dies during the policy term (*ie* where $T_{45} < 25$), and the second where the person is still alive at the maturity date (*ie* where $T_{45} \geq 25$). So the FLRV is:

$$L = \begin{cases} Sv^{T_{45}} - P\bar{a}_{\overline{T_{45}|}} & T_{45} < 25 \\ Sv^{25} - P\bar{a}_{\overline{25|}} & T_{45} \geq 25 \end{cases}$$

Thus,

$$L = Sv^{\min(T_{45},25)} - P\bar{a}_{\overline{\min(T_{45},25)|}}$$

1 Future Loss Random Variable - Example

Term insurance

An insurer issues a 25-year term insurance policy to life aged 45, with sum insured S payable immediately on death.
Level premiums are payable continuously. Expenses to be ignored. The net annual premium is P.

Write down the net future loss random variable, L, for this contract in terms of lifetime random variables for 45.

1 Future Loss Random Variable - Example

Term insurance

An insurer issues a 25-year term insurance policy to life aged 45, with sum insured *S* payable immediately on death.
Level premiums are payable continuously. Expenses to be ignored. The net annual premium is *P*.
Write down the net future loss random variable, *L*, for this contract in terms of lifetime random variables for 45.

This is same as endowment insurance, except that the sum assured is now only payable on death during the *n* – year term (there is no payment made on survival).

Thus,

$$L = \begin{cases} Sv^{T_{45}} - P\bar{a}_{\overline{T_{45}|}} & T_{45} < 25 \\ 0 - P\bar{a}_{\overline{25|}} & T_{45} \geq 25 \end{cases}$$

2 The Principle of Equivalence

Under the equivalence principle, the net premium is set such that the expected value of the future loss is zero at the start of the contract. It means that

$$\mathbf{E[L_0] = 0}$$

which implies that

E[PV of benefit outgo − PV of net premium income] = 0.

Thus, under the equivalence premium principle,

EPV of benefit outgo = EPV of net premium income

The net premium for a contract, given suitable mortality, interest and distribution assumptions would be found from the equation of expected present value.

2 Example

Consider an endowment insurance with term *n* years and sum insured *S* payable at the earlier of the end of the year of death or at maturity, issued to a select life aged *x*. Premiums of amount *P* are payable annually throughout the term of the insurance.

Derive expressions in terms of *S*, *P* and standard actuarial functions for

(a) the net future loss, L

(b) the mean of *l*,

(c) the annual net premium for the contract.

2 Example

(a) The future loss random variable is:

$$L = Sv^{\min(K_{[x]}+1,n)} - P\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|}$$

(b) The mean of L is:

$$E[L] = S\,E[v^{\min(K_{[x]}+1,n)}] - P\,E[\ddot{a}_{\overline{\min(K_{[x]}+1,n)}|}]$$

$$= S\,A_{[x]:\overline{n}|} - P\ddot{a}_{[x]:\overline{n}|}$$

(c) Setting the EPVs of the premiums and benefits to be equal gives the net premium as

$$P = \frac{S\,A_{[x]:\overline{n}|}}{\ddot{a}_{[x]:\overline{n}|}}$$

Question

An insurer issues a regular premium deferred annuity contract to a life aged x. Premiums are payable annually throughout the deferred period. The annuity benefit of X per year is payable annually in advance from age $x + n$ for the remainder of the life of (x).

(a) Write down the net future loss random variable in terms of lifetime random variables for x.

(b) Derive an expression for the net premium.

Solution

(a)

Let P denote the net premium.

Then

$$L = \begin{cases} 0 - P\,\ddot{a}_{\overline{K_{[x]}+1}|} & T_x \le n \\ Xv^n\ \ddot{a}_{\overline{K_{[x]}+1+n}|} - P\ddot{a}_{\overline{n}|} & T_x > n \end{cases}$$

(b)

EPV of annuity benefit is: $Xv^n\ {}_np_x\ \ddot{a}_{x+n}$

EPV of premiums: $P\ \ddot{a}_{x:\overline{n}|}$

By equating these EPVs we obtain the premium equation which gives

$$P = \frac{Xv^n\ {}_np_x\ \ddot{a}_{x+n}}{\ddot{a}_{x:\overline{n}|}}$$

Question

Consider a 10-year annual premium term insurance issued to a select life aged 50, with sum insured \$100 000 payable at the end of the year of death.

(a) Write down an expression for the net future loss random variable.

(b) Calculate the net annual premium.

Assume that mortality follows the Standard Select Survival Model, that interest is at 5% per year effective.

3 Calculating premiums that satisfy probabilities

Premiums (and reserves) can be calculated which satisfy probabilities involving the net future loss random variable.

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming i% per annum interest, AM92 Ultimate mortality, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

Let L be in FLRV and P be the premium.
Thus in such a case we need to find the smallest value of *P* such that:

$Pr(L > 0) \leq 0.05$

ie such that $Pr(L \leq 0) \geq 0.95$

4 Reserves

An important part of an Actuaries' work is that of reserving. Every life insurance office holds reserves.

What are reserves and what role do they play?
Why is it necessary for life offices to hold reserves?

4.1 Why hold Reserves?

A life insurance contract may be funded by level premiums, but the cost of mortality usually increases as the insured ages. As a result, the insured overpays in the earlier years and underpays in the later years. The company must save the overpayments of the early years in order to pay the benefits of the later years.

To put it a different way, if it is now *k* years since an insurance contract was sold to (x), the expected present value of the future premiums is usually less than the expected present value of the future benefits. The deficit must be made up with funds currently at hand. Such funds are a liability to the insurance company. The insurance company cannot regard these funds as its own, and cannot distribute them to its shareholders.

If it does so, then in the later years it may not have sufficient funds to pay for the excess of the cost of benefits over the premiums received. Thus, the company sets up reserves to ensure (as far as possible) that this does not happen, and to remain solvent.

4.2 Prospective Reserves

The prospective reserve for a life insurance contract which is in force (that is, has been written but has not yet expired through claim or reaching the end of the term) is defined to be, for a given basis:

The expected present value of the future outgo
less
the expected present value of the future income

This is the **prospective reserve because it looks forward to the future cash flows of the contract**. The prospective reserve is important because if the company holds funds equal to the reserve, and the future experience follows the reserve basis, then, averaging over many policies, the combination of reserve and future income will be sufficient to pay the future liabilities.

The reserve, therefore, gives the office a measure of the minimum funds it needs to hold at any point during the term of a contract. **The process of calculating a reserve is called the valuation of the policy**.

4.2 Calculating Net premium Prospective Reserves

The net premium reserve is the prospective reserve, where we make no allowance for future expenses, and where the premium used in the calculation is a notional net premium. This net premium is calculated using the equivalence principle and using the same assumptions as the reserve basis, and again making no allowance for future expenses.

The expression for the net future loss at policy duration *t* can be used to determine the net premium prospective reserve at policy duration *t*. This is done by determining the expected value of the gross future loss random variable.

In general we use the notation ${}_tV$ to represent the reserve held at policy duration *t*.

4.2 Calculating Net premium Prospective Reserves

Example: Whole Life Insurance

Consider a whole life assurance contract that has:

- a sum assured of S payable immediately on death
- annual premiums of P payable for the duration of the contract
- age at entry is x

The net future loss random variable after exactly t years is

$$L_t = Sv^{T_{x+t}} - P\ddot{a}_{\overline{K_{x+t}+1|}}$$

The prospective reserve after exactly t years is therefore

$$\begin{aligned} {}_tV &= E[L_t] \\ &= S\bar{A}_{x+t} - P\ddot{a}_{x+t} \end{aligned}$$

4.3 Reserve Conventions

- We often calculate reserves at integer durations, *ie* whole years. In this case, we calculate the reserve just before any payment of premium due on that date, and just after any payment of benefit payable in arrears due on that date.

- The general rule is, for valuation on the t th policy anniversary, payments in respect of the year $t - 1$ to t payable in arrears (*ie* on the t th anniversary) are assumed to have been paid, payments in respect of the year t to $t + 1$ payable in advance (and so are also due on the t th anniversary) are assumed not yet to have been paid.

4.4 Notations

- ${}_tV_x$ whole life assurance, pays at end of year of death, premium payable annually in advance

- ${}_tV_{x:\overline{n|}}$ endowment assurance with a term of n years ($t < n$) , pays at end of year of death or at end of term, premium payable annually in advance

- ${}_t\bar{V}_x$ whole life assurance, pays immediately on death, premium payable continuously

- ${}_t\bar{V}_{x:\overline{n|}}$ endowment assurance with a term of n years ($t < n$) , pays immediately on death or at end of term, premium payable continuously.

In each case listed above, the sum assured is assumed to be 1.

Question

Consider a 20-year endowment policy purchased by a select life aged 50. Level premiums are payable annually throughout the term of the policy and the sum insured, \$500 000, is payable at the end of the year of death or at the
end of the term, whichever is sooner.

The basis used by the insurance company for all calculations is the Standard Select Survival Model, 5% per year interest and no allowance for expenses.

(a) Show that the annual net premium, P, calculated using the equivalence principle, is \$15 114.33.

(b) Calculate tV_x = E$[L_t]$ for t = 10 and t = 11, in both cases just before the premium due at time t is paid.

Solution

Solution 7.1 (a) You should check that the following values are correct for this survival model at 5% per year interest:

$$\ddot{a}_{[50]:\overline{20}|} = 12.8456 \quad \text{and} \quad A_{[50]:\overline{20}|} = 0.38830.$$

The equation of value for P is

$$P\ddot{a}_{[50]:\overline{20}|} - 500\,000\,A_{[50]:\overline{20}|} = 0, \tag{7.1}$$

giving

$$P = \frac{500\,000\,A_{[50]:\overline{20}|}}{\ddot{a}_{[50]:\overline{20}|}} = \$15\,114.33.$$

Solution

(b) L^n_{10} is the present value of the future net loss 10 years after the policy was purchased, *assuming the policyholder is still alive at that time*. The policyholder will then be aged 60 and the select period for the survival model, two years, will have expired eight years ago. The present value at that time of the future benefits is $500\,000\, v^{\min(K_{60}+1,10)}$ and the present value of the future premiums is $P\,\ddot{a}_{\overline{\min(K_{60}+1,10)}|}$. Hence, the formulae for L^n_{10} and L^n_{11} are

$$L^n_{10} = 500\,000\, v^{\min(K_{60}+1,10)} - P\,\ddot{a}_{\overline{\min(K_{60}+1,10)}|}$$

and

$$L^n_{11} = 500\,000\, v^{\min(K_{61}+1,9)} - P\,\ddot{a}_{\overline{\min(K_{61}+1,9)}|}.$$

Taking expectations and using the annuity values

$$\ddot{a}_{60:\overline{10}|} = 7.9555 \quad \text{and} \quad \ddot{a}_{61:\overline{9}|} = 7.3282$$

we have

$$\mathrm{E}[L^n_{10}] = 500\,000 A_{60:\overline{10}|} - P\ddot{a}_{60:\overline{10}|} = \$190\,339$$

and

$$E[L^n_{11}] = 500\,000 A_{61:\overline{9}|} - P\ddot{a}_{61:\overline{9}|} = \$214\,757.$$

4.5 Retrospective Reserves

The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

The accumulated value allowing for interest and survivorship of the premiums received to date
less
the accumulated value allowing for interest and survivorship of the benefits paid to date

The retrospective reserve on a given basis tells us how much the premiums less claims have accumulated to, averaging over a large number of policies**.**

4.5 Retrospective Accumulation

The basic idea is that we consider a group of lives, who are regarded as identical and stochastically independent as far as mortality is concerned.

At age x , each life transacts an identical life insurance contract. Under these contracts, payments will be made (the direction of the payments is immaterial), depending on the experience of the members of the group.

We imagine these payments being accumulated in a fund at rate of interest i .

After n years, we divide this fund equally among the surviving members of the group. (If the fund is negative we imagine charging the survivors in equal shares.)

The retrospective accumulation is defined as the amount that each survivor would receive, as the group size tends towards infinity.

4.5 Retrospective Accumulation

Suppose that there are L_n survivors at age $x + n$ out of L 'starters' at age x, and that the accumulated fund at age $x + n$ is $F_n(L)$. The retrospective accumulation of the benefit under consideration is defined to be:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n}$$

Clearly L_n and $F_n(L)$ are random variables. The process of taking the limit eliminates the awkward possibility that $L_n = 0$, but also since:

$$\lim_{L \to \infty} \frac{F_n(L)}{L_n} = E[F_n(1)] \quad \text{And} \quad \lim_{L \to \infty} \frac{L_n}{L} = npx$$

by the law of large numbers:

$$\boldsymbol{\lim_{L \to \infty} \frac{F_n(L)}{L_n} = \frac{E[F_n(1)]}{npx}}$$

4.5 Retrospective Accumulation

Example: Term assurance

Consider a term assurance that pays 1 at the end of the year of death occurring within n years. Now:

$$F_n(1) \begin{cases} = (1+i)^{n-(Kx+1)} & \text{if } K_x < n \\ = 0 & \text{if } K_x \geq n \end{cases}$$

So

$$E[F_n(1)] = \sum_0^{n-1} (1+i)^{n-(K+1)}\, k|\, q_x = (1+i)^n A^1_{x:\,\overline{n|}}$$

Hence the accumulation of the term assurance benefit is

$$\frac{(1+i)^n A^1_{x:\,\overline{n|}}}{np_x}$$

4.5 Retrospective Accumulation

Example: Temporary Annuity-due

Consider a temporary annuity-due paying 1 per annum with a term of *n* years. $F_n(1)$ has the following distribution:

$$F_n(1) \begin{cases} = (1+i)^{n-(Kx+1)}\ddot{s}_{\overline{Kx+1|}} & \text{if } K_x < n \\ = \ddot{s}_{\overline{n|}} & \text{if } K_x \geq n \end{cases}$$

Recall that $\ddot{s}_{\overline{n|}}$ is the accumulated value of certain payments of 1 *pa*, payable annually in advance for *n* years, accumulated to the end of the *n* years.

Hence:

$$E[F_n(1)] = \sum_0^{n-1}(1+i)^{n-(K+1)}\ddot{s}_{\overline{Kx+1|}}\ k|\,q_x + \ddot{s}_{\overline{n|}}npx$$
$$= (1+i)^n\ \ddot{a}_{x:\overline{n|}}$$

Therefore the accumulation of the temporary annuity-due is:

$$\ddot{s}_{x:\overline{n|}} = \frac{(1+i)^n\ \ddot{a}_{x:\overline{n|}}}{npx}$$

4.5 Retrospective Accumulation

In General, we can write down the retrospective accumulation of any benefit after n years in the same way – we multiply its EPV by $\frac{(1+i)^n}{{}_np_x}$.

4.5 Net Premium Retrospective Reserve

A generic definition of a gross premium retrospective reserve is:

Retrospective accumulation of past premiums – Retrospective accumulation of past benefits

4.5 Net Premium Retrospective Reserve

Example: Whole Life Insurance

The net premium retrospective reserve at policy duration t is:

$$\frac{l_x}{l_{x+t}}(1+i)^t\{\, P\ddot{a}_{x:\,\overline{t}|} - S\,\bar{A}^{1}_{x:\,\overline{t}|} \,\}$$

The expression in brackets gives us 'the expected present value at time 0 of the premiums less benefits and expenses payable in the first t policy years'. The factors outside the brackets then translate that into a value at time t .

5 Equality of prospective and retrospective reserves

Conditions for equality

If:
1. the retrospective and prospective reserves are calculated on the same basis, and
2. this basis is the same as the basis used to calculate the premiums used in the reserve calculation,

then the retrospective reserve will be equal to the prospective reserve

5 Equality of prospective and retrospective reserves

Example: Whole Life Insurance

The net premium equation is

$$P\,\ddot{a}_x = S\,\bar{A}_x$$

Split the premium equation up at time t

$$P(\ddot{a}_{x:\overline{t|}} + v^t\,{}_tp_x\,\ddot{a}_{x+t}) = S(\bar{A}^1_{x:\overline{t|}} + v^t\,{}_tp_x\,\bar{A}_{x+t})$$

Rearrange as follows:

$$P\,\ddot{a}_{x:\overline{t|}} - S\,\bar{A}^1_{x:\overline{t|}} = v^t\,{}_tp_x\,(P\ddot{a}_{x+t} - S\bar{A}_{x+t})$$

Dividing both sides by $v^t\,{}_tp_x$ then gives

$$\boldsymbol{(P\,\ddot{a}_{x:\overline{t|}} - S\,\bar{A}^1_{x:\overline{t|}})\,\frac{(1+i)^t}{{}_tp_x} = P\ddot{a}_{x+t} - S\bar{A}_{x+t}}$$

The expression on the left-hand side of this equation is the net premium retrospective reserve at time t and the expression on the right is the net premium prospective reserve at time t .

6 Recursive relationship between reserves for annual premium contracts

If the expected cash flows (*ie* premiums and benefits) during the policy year (t, t + 1) are evaluated and allowance is made for the time value of money, we can develop a recursive relationship linking net premium policy values in successive years.

We illustrate this using a whole life assurance secured by level annual premiums, but the method extends to all standard contracts.

Example: Whole life assurance
Net premium policy value at duration t = ${}_tV$
Premium paid at t = P
Expected claims paid at t +1 = q_{x+t} S
Net premium policy value at duration t + 1 = ${}_{t+1}V$

Then the equation of value at time t + 1 for these cash flows is:

$$({}_tV + P)(1+i) - q_{x+t}\,S = (1 - q_{x+t})\,{}_{t+1}V$$

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6 Recursive relationship between reserves for annual premium contracts

The equation of value at time $t + 1$ for these cash flows is:

$({}_tV + P)(1+i) - q_{x+t}\,S = (1 - q_{x+t})\ {}_{t+1}V$

Equation above will only be satisfied if all quantities are calculated on mutually consistent bases: *ie* using the same interest, mortality assumptions for the reserves, premium and experience over the year.

Dividing through by $p_{x+t} = 1 - q_{x+t}$, we obtain a formula for calculating the reserve at the end of the year from the reserve at the beginning of the year:

$$\frac{({}_tV + P)(1+i) - q_{x+t}\,S}{p_{x+t}} = {}_{t+1}V$$

The equation gives a recursive relationship between policy values in successive years.

Question

Consider a 20-year endowment policy purchased by a select life aged 50. Level premiums of $23 500 per year are payable annually throughout the term of the policy. A sum insured of $700 000 is payable at the end of the term if the life survives to age 70. On death before age 70 a sum insured is payable at the end of the year of death equal to the policy value at the start of the year in which the policyholder dies.

The policy value basis used by the insurance company is as follows:
Survival model: Standard Select Survival Model
Interest: 3.5% per year
Expenses: nil

Calculate ${}_{15}V$, the policy value for a policy in force at the start of the 16th year.

Solution

Using, the recursive relationship, we have:

$$({}_tV + P) \times 1.035 = q_{[50]+t}\, S_{t+1} + p_{[50]+t}\; {}_{t+1}V \quad \text{for } t = 0, 1, \ldots, 19,$$

where $P = \$23\,500$. For the final year of this policy, the death benefit payable at the end of the year is ${}_{19}V$ and the survival benefit is the sum insured, \$700 000. Putting $t = 19$ in the above equation gives:

$$({}_{19}V + P) \times 1.035 = q_{69}\; {}_{19}V + p_{69} \times 700\,000.$$

Tidying this up and noting that $S_{t+1} = {}_tV$, we can work backwards as follows:

$$\begin{aligned}
{}_{19}V &= (p_{69} \times 700\,000 - 1.035P)/(1.035 - q_{69}) = 652\,401,\\
{}_{18}V &= (p_{68} \times {}_{19}V - 1.035P)/(1.035 - q_{68}) = 606\,471,\\
{}_{17}V &= (p_{67} \times {}_{18}V - 1.035P)/(1.035 - q_{67}) = 562\,145,\\
{}_{16}V &= (p_{66} \times {}_{17}V - 1.035P)/(1.035 - q_{66}) = 519\,362,\\
{}_{15}V &= (p_{65} \times {}_{16}V - 1.035P)/(1.035 - q_{65}) = 478\,063.
\end{aligned}$$

Hence, the answer is \$478 063. □

7 A special result for the net premium reserve for some endowment and whole life assurance contracts

A special result for the net premium reserve for some endowment and whole life assurance contracts follows from the fact that the net premium is calculated on the same basis as the reserve basis.

Example: Whole life assurance

${}_tV_x$ is the net premium reserve at duration t for a whole life assurance policy, with sum assured of 1 payable at the end of the year of death, and with level annual premiums payable during the duration of the policy. The net premium P_x for this contract is: $P_x = \frac{A_x}{\ddot{a}_x}$

and so:

$${}_tV_x = A_{x+t} - \mathrm{P}\ddot{a}_{x+t} = A_{x+t} - \frac{A_x}{\ddot{a}_x}\ddot{a}_{x+t}$$

$$= (1 - \mathrm{d}\ddot{a}_{x+t}) - \frac{(1 - \mathrm{d}\ddot{a}_x)}{\ddot{a}_x}\ddot{a}_{x+t}$$

$$= 1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}$$

The net premium reserve for the above contract with a sum assured of S would then be:

$$S\,{}_tV_x = \mathrm{S}\left(1 - \frac{\ddot{a}_{x+t}}{\ddot{a}_x}\right)$$

8 Mortality Profit on a Single Policy

8.1 Death strain at risk (DSAR)

Consider a policy issued t years ago to a life then aged x, with sum assured S payable at the end of the year of death. Also, assume that no survival benefit is due if the life survives to $t+1$.
Let ${}_tV$ be the reserve at time t. Then we define the *death strain* in the policy year t to $t+1$ to be the random variable, DS, say,

$$DS = \begin{cases} 0 & \text{if the life survives to } t+1 \\ (S - {}_{t+1}V) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

The maximum death strain, (S - ${}_{t+1}$V) is called the death strain at risk or DSAR.

8 Mortality Profit on a Single Policy

What do you think is the intuitive understanding or reasoning behind Death strain at risk (DSAR)?

8 Mortality Profit on a Single Policy

Death strain at risk (DSAR)

The word strain is used loosely to mean a cost to the company. The reasoning behind the DSAR definition is seen more clearly if we rearrange the recursive relationship between ${}_tV$ and ${}_{t+1}V$.

Assuming level net premiums, the general recursive relationship between reserves ${}_tV$ and ${}_{t+1}V$ is:

$$\begin{aligned}({}_tV + P)(1+i) &= q_{x+t}\, S + (1 - q_{x+t})\, {}_{t+1}V \\ &= {}_{t+1}V + q_{x+t}\,(S - {}_{t+1}V)\end{aligned}$$

In words, the reasoning is that for each policy we must pay out at least ${}_{t+1}V$ at the end of the year. In addition, if the policy becomes a claim during the year, with probability q_{x+t}, then we must pay out an extra sum of (S - ${}_{t+1}V$) which is the DSAR.

Note that q_{x+t} is the probability of dying in the year t to $t + 1$, and therefore $x + t$ is the age at the *start* of the year.

8 Mortality Profit on a Single Policy

8.2 Expected Death strain [EDS]

The expected amount of the death strain is called the *expected death strain* (*EDS*). This is the amount that the life insurance company expects to pay in addition to the year-end reserve for the policy.

The probability of claiming in the policy year t to $t + 1$ is q_{x+t} so that:

$$\mathbf{EDS} = q_{x+t}\,(S - {}_{t+1}V)$$

8 Mortality Profit on a Single Policy

8.3 Actual Death strain [ADS]

The actual death strain is simply the observed value at t 1 of the death strain random variable, that is:

$$\text{ADS} = \begin{cases} 0 & \text{if the life survives to } t+1 \\ (S - {}_{t+1}V) & \text{if the life dies in the year } [t, t+1) \end{cases}$$

8 Mortality Profit on a Single Policy

8.4 Mortality Profit

The mortality profit is defined as:

Mortality profit = Expected Death Strain – Actual Death Strain

- The EDS is the amount the company *expects* to pay out, in addition to the year-end reserve for a policy.
- The ADS is the amount it *actually* pays out, in addition to the year-end reserve.
- If it actually pays out less than it expected to pay, there will be a profit. If the actual strain is greater than the expected strain, there will be a loss

8.5 Mortality Profit on a Portfolio of policies

We are often interested in analysing the experience of a group of similar policies. We use the term portfolio of policies to mean any group of policies. In this case we simply sum the EDS and the ADS over all the relevant policies. If all lives are the same age, and subject to the same mortality table, this gives:

Total DSAR = $\sum_{\text{all policies}} (S - {}_{t+1}V)$

$$\begin{aligned}\text{Total EDS} &= \sum_{\text{all policies}} q_{x+t} (S - {}_{t+1}V) \\ &= q_{x+t} \sum_{\text{all policies}} (S - {}_{t+1}V) \\ &= q_{x+t} \text{ X Total DSAR}\end{aligned}$$

Total ADS = $\sum_{\text{death claims}} (S - {}_{t+1}V)$

Mortality Profit = Total EDS – Total ADS

8.6 Allowing for death benefits payable immediately

Where death benefits are payable immediately on death, in the calculation of the death strain we allow for interest between the time of payment and the end of the year of death. In this case, the death strain defined earlier would become

$$\textbf{DS} = \begin{cases} \textbf{0} & \textbf{if the life survives to t + 1} \\ \left(\textbf{S}(1+i)^{1/2} - {}_{t+1}V \right) & \textbf{if the life dies in the year [t, t+1)} \end{cases}$$

Similar adjustments can be applied to EDS and ADS.

8.7 Allowing for Survival benefits

Suppose the contract provides for a benefit at the end of policy year t to $t + 1$. By convention, the expected present value of this will have been included in ${}_tV$ but will fall outside the computation of ${}_{t+1}V$. So, the survival benefit needs to be allowed for as an additional payment.

Let R be the benefit payable at the end of the policy year t to $t + 1$ contingent on the survival of the policyholder. Assuming death benefits are paid at the end of the year of death, the recursive relationship between successive reserves is now

$$\begin{aligned}({}_tV + P)(1+i) &= S\, q_{x+t} + (1 - q_{x+t})\,({}_{t+1}V + R) \\ &= {}_{t+1}V + R + q_{x+t}(S - ({}_{t+1}V + R))\end{aligned}$$

The DS is now:

$$\mathbf{DS} = \begin{cases} 0 & \textbf{if the life survives to t + 1} \\ (S - ({}_{t+1}V + R)) & \textbf{if the life dies in the year [t, t+1)} \end{cases}$$

8.7 Allowing for Survival benefits

- The expected death strain is then $\boldsymbol{q_{x+t}}$ **(S – ($_{t+1}V$ + R))**

- The actual death strain is **0 if the life survived** the year and **(S – ($_{t+1}V$ + R)) if the life died** during the year

- The **mortality profit is EDS - ADS**, as before.

8.8 Annuities

- In the case of an annuity of *R pa*, payable annually in arrears, with no death benefit, the DSAR would be – (${}_{t+1}V$ + R). In this case each death causes a negative strain or *release of reserves*.

- In the case of an annuity of *R pa*, payable annually in advance, with no death benefit, the DSAR would be – ${}_{t+1}V$ only. The annuity payment is made by all policies in force at the start of the year, and is not affected by whether or not the policyholder survives the year.

Question

CT5 April 2010
On 1 January 2005, a life insurance company issued 1,000 10-year term assurance policies to lives aged 55 exact. For each policy, the sum assured is £50,000 for the first five years and £25,000 thereafter. The sum assured is payable immediately on death and level annual premiums are payable in advance throughout the term of this policy or until earlier death.

The company uses the following basis for calculating premiums and reserves:
Mortality AM92 Select
Interest 4% per annum
Expenses Nil

(i) Calculate the net premium retrospective reserve per policy as at 31 December 2009. [6]

(ii) (a) Give an explanation of your numerical answer to (i) above.
(b) Describe the main disadvantage to the insurance company of issuing this policy.
(c) Give examples of how the terms of the policy could be altered so as to remove this disadvantage. [3]

There were, in total, 20 deaths during the years 2005 to 2008 inclusive and a further 8 deaths in 2009.
(iii) Calculate the total mortality profit or loss to the company during 2009. [3]
[Total 12]

Solution

(i) Annual premium P for the term assurance policy is given by:

$$P = \frac{25,000\bar{A}^{1}_{[55]:\overline{10}|} + 25,000\bar{A}^{1}_{[55]:\overline{5}|}}{\ddot{a}_{[55]:\overline{10}|}}$$

where

$$25,000\bar{A}^{1}_{[55]:\overline{10}|} + 25,000\bar{A}^{1}_{[55]:\overline{5}|}$$

$$= 25,000 \times (1+i)^{1/2} \times \left((A_{[55]} - v^{10}\,{}_{10}p_{[55]}A_{65}) + (A_{[55]} - v^{5}\,{}_{5}p_{[55]}A_{60}) \right)$$

$$= 25,000 \times 1.019804 \times \begin{pmatrix} (0.38879 - 0.67556 \times \dfrac{8821.2612}{9545.9929} \times 0.52786) \\ +(0.38879 - 0.82193 \times \dfrac{9287.2164}{9545.9929} \times 0.4564) \end{pmatrix}$$

$$= 25,495.10 \times \left((0.38879 - 0.32953) + (0.38879 - 0.36496) \right) = 2118.39$$

Therefore

$$P = \frac{2118.39}{8.228} = 257.46$$

Solution

Net Premium Retrospective Reserves at the end of the fifth policy year is given by:

$$(1+i)^5 \times \frac{l_{[55]}}{l_{60}} \times \left[P\ddot{a}_{[55]:\overline{5}|} - 50,000\bar{A}^{\,1}_{[55]:\overline{5}|} \right]$$

$$= 1.21665 \times \frac{9545.9929}{9287.2164} \times [257.46 \times 4.59 - 50,000 \times 1.019804 \times (0.38879 - 0.36496)]$$

$$= -41.71$$

Solution

(ii) **Explanation** – more cover provided in the first 5 years than is paid for by the premiums in those years. Hence policyholder “in debt” at time 5, with size of debt equal to negative reserve.

Disadvantage – if policy lapsed during the first 5 years (and possibly longer), the company will suffer a loss which is not possible to recover from the policyholder.

Possible alterations to policy structure

Collect premiums more quickly by shortening premium payment term or make premiums larger in earlier years, smaller in later years

Change the pattern of benefits to reduce benefits in first 5 years and increase them in last 5 years.

Solution

(iii) Mortality Profit = EDS – ADS

Death strain at risk = 50,000 – (–42) = 50,042

$$EDS = (1000 - 20) \times q_{59} \times 50,042$$
$$= 980 \times 0.00714 \times 50,042 = 350,154$$

$$ADS = 8 \times 50,042 = 400,336$$

Total Mortality Profit = 350,154 – 400,336 = -£50,182 (i.e. a mortality loss)

Question

CT5 April 2008

A life assurance company issues the following policies:

- 10-year term assurances with a sum assured of £50,000 where the death benefit is payable at the end of the policy year of death
- 10-year pure endowment assurances with a sum assured of £50,000 payable on maturity

For the term assurance and pure endowment policies, premiums are paid annually in advance.

The company sold 5,000 policies of each type to lives then aged 50 exact. During the first policy year, there were five actual deaths from each of the two types of policies written.

(i) Assuming each type of policy was sold to a distinct set of lives (i.e. no life buys more than one type of policy).
(a) Calculate the death strain at risk for each type of policy at the end of the second policy year of the policies.
(b) During the second policy year, there were ten deaths from each of the two types of policy written. Calculate the total mortality profit or loss to the company during the second policy year.

Basis:
Interest 4% per annum
Mortality AM92 Ultimate for term assurance and pure endowment
Expenses – Nil

Question

Continued

(ii) The company now discovers that 5,000 lives had bought one of each type of policy.
(a) State whether the mortality profit or loss calculated would now be higher, lower or unchanged to that calculated in (i)(b).

(b) State whether the variance of the benefits paid out by the company in future years would be higher, lower or unchanged to that in (i). Explain your answer by general reasoning.

Solution

(i) (a) Annual premium for pure endowment with £50,000 sum assured given by:

$$P^{PE} = \frac{50,000}{\ddot{a}_{50:\overline{10|}}} \times {}_{10}p_{50} \times v^{10} = \frac{50,000}{8.314} \times 0.64601 = 3885.10$$

Annual premium for term assurance with £50,000 sum assured given by:

$$P^{TA} = P^{EA} - P^{PE} = \frac{50,000 A_{50:\overline{10|}}}{\ddot{a}_{50:\overline{10|}}} - P^{PE}$$

$$= \frac{50,000 \times 0.68024}{8.314} - 3885.10 = 205.83$$

Solution

Reserves at the end of the second year:

for pure endowment with £50,000 sum assured given by:

$${}_2V^{PE} = 50{,}000 \times {}_8p_{52} \times v^8 - P^{PE}\ddot{a}_{52:\overline{8}|}$$

$$= 50{,}000 \times 0.70246 - 3885.10 \times 6.910 = 35123.0 - 26846.04 = 8276.96$$

for term assurance with £50,000 sum assured given by:

$${}_2V^{TA} = {}_2V^{EA} - {}_2V^{PE}$$

$$= 50{,}000 A_{52:\overline{8}|} - (3885.1 + 205.83)\ddot{a}_{52:\overline{8}|} - 8276.96$$

$$= 50{,}000 \times 0.73424 - 4090.93 \times 6.91 - 8276.96$$

$$= 166.71$$

Solution

Sums at risk:

Pure endowment:	DSAR = 0 – 8,276.96 = –8,276.96
Term assurance:	DSAR = 50,000 – 166.71 = 49,833.29

Solution

(b) Mortality profit = EDS – ADS

For term assurance

$$EDS = 4995 \times q_{51} \times 49,833.29 = 4995 \times .002809 \times 49,833.29 = 699,208.65$$

$$ADS = 10 \times 49,833.29 = 498,332.90$$

mortality profit = 200,875.75

For pure endowment

$$EDS = 4995 \times q_{51} \times -8,276.96 = 4995 \times .002809 \times -8,276.96 = -116,133.65$$

$$ADS = 10 \times -8,276.96 = -82,769.60$$

mortality profit = –33,364.05

Hence, total mortality profit = £167,511.70

Solution

(ii) (a) The actual mortality profit would remain as that calculated in (i) (b).

(b) The variance of the benefits would be lower than that calculated in (i).

In this case, the company would not pay out benefits under both the PE and the TA but will definitely pay out one of the benefits. Under the scenario in (i), the company could pay out all the benefits (if all the TA policyholders die and the PE policyholders survive). Alternatively, they could pay out no benefits at all (if all the TA policyholders survive and the PE policyholders immediately die).

Question – HW

CT5 September 2012
On 1 January 2007, a life insurance company sold a large number of 30-year pure endowment policies to lives then aged 35 exact. The sum assured under each policy is £125,000 payable on maturity. Premiums are payable annually in advance
throughout the term of the policy.

There were 3521 pure endowment policies still in force on 1 January 2011 and 8 policyholders died during 2011.

Calculate the total mortality profit or loss to the life insurance company during 2011 assuming the company calculates net premium reserves on the following basis:

Mortality AM92 Select
Interest 4% per annum
Expenses Nil

Question – HW

CT5 September 2016 Q10
A 25-year “double” endowment assurance policy is issued to a group of lives aged 40 exact. Each policy provides a sum assured of 25,000 payable at the end of the year of death or 50,000 payable if the life survives until the maturity date.
Premiums are payable annually in advance throughout the term of the policy or until earlier death.

The following information has been provided:
Number of deaths during the 17th policy year: 24
Number of policies in force at the end of the 17th policy year: 5,350

(i) Calculate, showing all your workings, the profit or loss for the group arising from mortality in the 17th policy year.
[7]

(ii) Comment on your result. [2]

Basis:
Mortality AM92 Select
Rate of interest 4% per annum
Expenses Ignore