IAQS INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc Semester 3

Subject : Pricing and Reserving for Life Insurance Products

Chapter: Unit 4

Chapter Name: Multiple life models

Today's Agenda

0. Multiple lives - Introduction

1. Joint Life Functions

1. Last Survivor Functions

1. Present values involving two lives

1. Calculations

1. Contingent and Reversionary Assurances and Annuities

1. Annuities payable m times per year

0 Multiple lives - Motivation

Till now we have touched upon assurance functions, annuity functions, premiums and reserves for a single life.

But an **actuary sometimes has to deal with collections of lives, multiple lives**. The valuation of benefits and premiums for an insurance policy where the **payments depend on the survival or death of two lives.** Such policies are very common.

Some examples are:

- A whole life insurance policy may pay a benefit at the second death of a husband and a wife.

- A whole life annuity immediate may pay a fixed amount per month, such as 1000, to a set of two people (such as husband and *wife)* until they are both dead.

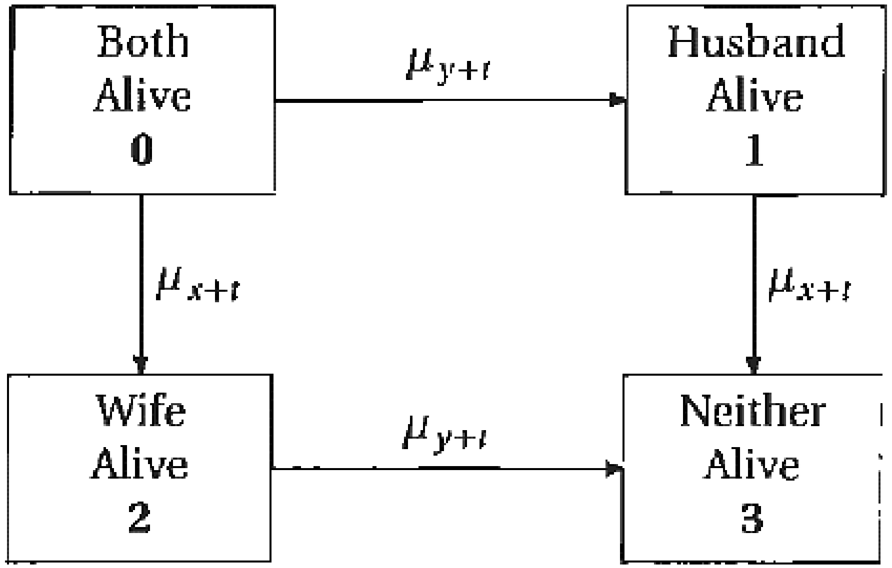
- Complicated estate provisions may involve several lives. For example, a will may call for the payment after death of the parent of 1000 per month to each of the children until they turn 18.

0 Multiple lives

We will only discuss a two-life model.

We use a double subscript on the forces: x is the age oft the husband and *y* is the age of the wife. Assuming that the two lives are independent, the survival of one life does not depend on whether the other one is alive or not.

This can be shown by the following Markov chain diagram:

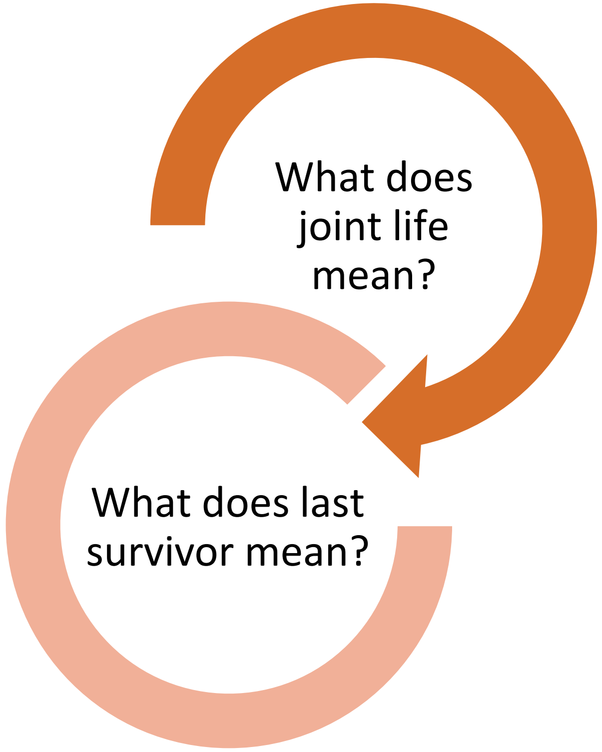


0 Introduction

Two lives can be **modelled using joint life functions or last survivor functions.**

This chapter covers the following topics:

- random variables describing basic joint life and last survivor functions
- determining simple joint life and last survivor probabilities
- present values of simple joint life and last survivor policies
- calculating premiums and reserves for policies based on two lives.



1 Joint life functions

We now consider annuity and assurance functions that depend upon the death or survival of two lives.

The random variables of interest are T_x and T_y , the future lifetimes of two lives, one aged x and the other aged y . Throughout the analysis of these problems, we assume that T_x and T_y are independent random variables.

The random variable T_{xy} measures the joint lifetime of (x) and (y) , *ie* the time while both (x) and (y) remain alive, which is the time until the first death of (x) and (y) .

We can write:

$$T_{xy} = \min[T_x, T_y]$$

1 Joint life functions

The **cumulative distribution function** of this random variable can be written:

$F_{T_{xy}}(t)$ = P[T_{xy} ≤ t]

which we write as:

$F_{T_{xy}}(t)$ = 1 - ${}_tp_{xy}$
= P[min{T_x, T_y} ≤ t]
= 1 – P[T_x > t and T_y > t]
= 1 – P[T_x > t] P[T_y > t] -> (since the random variables *Tx* and *Ty* are independent)

So:
$\boldsymbol{F_{T_{xy}}(t) = 1 - {}_tp_x \, {}_tp_y}$

1 Joint life status

The random variable T_{xy} is characterised by the *joint life* status xy, which is the status of both lives x and y being alive. If either one of x or y dies, the joint life status is said to *fail*.

The random variable T_{xy} represents the future time until the failure of the status, which in this case is the joint life status xy.

What does the probability notation ${}_tp_{xy}$ introduced above represent?

1 Joint life functions

The density function of T_{xy} can be obtained by differentiating the cumulative distribution function:

$$\begin{aligned} f_{T_{xy}}(t) &= \frac{d}{dt}[1 - {}_tp_x \, {}_tp_y] \\ &= - \frac{d}{dt} \, {}_tp_x \, {}_tp_y \\ &= - \, {}_tp_x \frac{d}{dt} {}_tp_y - {}_tp_y \frac{d}{dt} {}_tp_x \\ &= - \, {}_tp_x \, (- \, {}_tp_y \, \mu_{y+t}) - {}_tp_y \, (- \, {}_tp_x \, \mu_{x+t}) \end{aligned}$$

The last line above follows from the fact that :

$$f_{T_x}(t) = {}_tp_x \, \mu_{x+t} = \frac{d}{dt} F_{T_x}(t) = \frac{d}{dt} P(T_x \leq t) = \frac{d}{dt} {}_tq_x = \frac{d}{dt}(1 - {}_tp_x) = -\frac{d}{dt} {}_tp_x$$

Thus finally:

$$\boldsymbol{f_{T_{xy}}(t) = {}_tp_x \, {}_tp_y \, (\mu_{x+t} + \mu_{y+t})}$$

1 Joint life table functions

It is helpful to develop the joint life functions *lxy* , *dxy* and *qxy* to help in the numerical evaluation of expressions that are the solution to problems involving more than one life. We define these functions in terms of the single life functions.

- $_t p_{xy} = {_t p_x}\ {_t p_y}$

Using the independence assumption:

$$_t p_{xy} = \frac{l_{x+t}}{l_x} \cdot \frac{l_{y+t}}{l_y}$$

So we write:

- $l_{xy} = l_x\ l_y$
- And, $_t p_{xy} = \dfrac{l_{x+t:y+t}}{l_{xy}}$

only separating the subscripts with colons when the exact meaning of the function would be unclear if the colons were omitted.

- Then, $d_{xy} = l_{xy} - l_{x+1:y+1}$ and $q_{xy} = \dfrac{d_{xy}}{l_{xy}}$

1 Joint life table functions

The force of failure of the joint life status can be derived in the usual way:

$$\mu_{x+t:\,y+t} = -\frac{1}{l_{x+t:\,y+t}}\frac{d}{dt}\,l_{x+t:\,y+t}$$

This can then be related to the forces of mortality in the life tables for the single lives *x* and *y*:

$$\begin{aligned}\mu_{x+t:\,y+t} &= -\frac{d}{dt}\log_e l_{x+t:\,y+t}\\ &= -\frac{d}{dt}\log_e l_{x+t}\;l_{y+t}\\ &= -\frac{d}{dt}\{\log_e l_{x+t} + \log_e l_{y+t}\}\end{aligned}$$

Thus:

$$\boldsymbol{\mu_{x+t:\,y+t} = \mu_{x+t} + \mu_{y+t}}$$

So we can write:

$$\boldsymbol{f_{T_{xy}}(t) = {}_tp_{xy}\,\mu_{x+t:\,y+t}}$$

Question

You are given the following mortality table:

	Male	Female
x	q_x	q_x
80	0.10	0.07
81	0.12	0.09
82	0.14	0.11
83	0.16	0.13
84	0.18	0.15

Kevin and Kira are a couple. Kevin, a male, is age 82, and Kira, a female is age 80. Kevin and Kira have independent future lifetimes.

Calculate the probability that the first death for this couple will occur within the third year from the current date.

Solution

The symbol for what we're trying to calculate is ${}_{2|}q_{82:80}$. We calculate the probability that the joint status will survive two years, and then subtract the probability that joint status will survive three years.

For Kevin:
$$ {}_{2}p_{82} = (1-0.14)(1-0.16) = 0.7224 $$
$$ {}_{3}p_{82} = (0.7224)(1-0.18) = 0.592368 $$

For Kira:
$$ {}_{2}p_{80} = (1-0.07)(1-0.09) = 0.8463 $$
$$ {}_{3}p_{80} = (0.8463)(1-0.11) = 0.753207 $$

For the joint status, we therefore have:
$$ {}_{2}p_{82:80} = (0.7224)(0.8463) = 0.611367 $$
$$ {}_{3}p_{82:80} = (0.592368)(0.753207) = 0.446176 $$

So the probability we seek is

$$ {}_{2}p_{82:80} - {}_{3}p_{82:80} = 0.611367 - 0.446176 = \boxed{0.165191} $$

2 Last Survivor Functions

Sometimes we are interested in the time when the last life of a set of lives fails.

To study this situation, we define the ***last survivor* status** as **a status that fails only when every member of the status fails**.

State two examples of last survivor policies, i.e. (payment contingent on the death of the second life)

2 Last Survivor Functions

The random variable $T_{\overline{xy}}$ measures the time until the last death of (*x*) and (*y*) , *ie* the time while at least one of (*x*) and (*y*) remains alive.

We can write:

$T_{\overline{xy}} = \max[T_x, T_y]$

The cumulative distribution function of this random variable can be written:

$F_{T_{\overline{xy}}}(t) = \mathrm{P}[T_{\overline{xy}} \leq t] = {}_tq_{\overline{xy}}$

$= \mathrm{P}\,[\max\{T_x, T_y\} \leq t]$

$= \mathrm{P}\,[T_x \leq t \text{ and } T_y \leq t]$

$= \mathrm{P}\,[T_x \leq t]\,\mathrm{P}[T_y \leq t]$ -> (since the random variables *Tx* and *Ty* are independent)

$\boldsymbol{F_{T_{\overline{xy}}}(t) = {}_tq_x\, {}_tq_y}$

2 Last Survivor & Joint life Functions

Let's discuss relationships between joint status and last survivor status.

First of all, let T_{xy} and $T_{\overline{xy}}$ be the future lifetime random variables for the joint life and last survivor statuses respectively.

In a group of two lives, $T_x + T_y$ is the sum of the future lifetimes of *(x)* and *(y)*. Since one of them dies first and the other dies second, $T_{xy} + T_{\overline{xy}}$ is also the sum of the future Lifetimes *of (x)* and *(y)*.

Therefore:

$$T_x + T_y = T_{xy} + T_{\overline{xy}}$$

Thus:

$$\boldsymbol{T_{\overline{xy}} = T_x + T_y - T_{xy}}$$

2 Last Survivor & Joint life Functions

Relationship between joint status and last survivor status.

Important result

Function of single life status x + Function of single life status y − Function of joint life status xy = Function of last survivor status

2 Last Survivor Functions

The cumulative distribution function of this random variable can be written as:

$$
\begin{aligned}
F_{T_{\overline{xy}}}(t) &= (1 - {}_tp_x)(1 - {}_tp_y) \\
&= 1 - {}_tp_x - {}_tp_y + {}_tp_x {}_tp_y \\
&= (1 - {}_tp_x) + (1 - {}_tp_y) - (1 - {}_tp_x {}_tp_y) \\
&= F_{T_x}(t) + F_{T_y}(t) - F_{T_{xy}}(t)
\end{aligned}
$$

So we have:

$$\mathbf{{}_tq_{\overline{xy}} = {}_tq_x + {}_tq_y - {}_tq_{xy}}$$

2 Last Survivor Functions

The density function of $T_{\overline{xy}}$ can be obtained by differentiating the cumulative distribution function:

$$f_{T_{\overline{xy}}}(t) = \frac{d}{dt}[1 - {}_tp_x - {}_tp_y + {}_tp_x\,{}_tp_y]$$

$$= {}_tp_x\, \mu_{x+t} + {}_tp_y\, \mu_{y+t} - {}_tp_{xy}\, \mu_{x+t:y+t}$$

$$= f_{T_x}(t) + f_{T_y}(t) - f_{T_{xy}}(t)$$

Tip

- For joint life statuses, it is often easier to work with p -type (survival) functions, since
  $\boldsymbol{{}_t p_{xy} = {}_t p_x \, {}_t p_y}$

- For last survivor statuses, it is often easier to work with q -type (mortality) functions, since
  $\boldsymbol{{}_t q_{\overline{xy}} = {}_t q_x \, {}_t q_y}$

Question

Mortality rates for two lives (x) and *(y)* are as follows:

t	q_{x+t}	q_{y+t}
0	0.01	0.02
1	0.02	0.03
2	0.03	0.04

Future lifetimes for the two lives are independent.

i) Calculate ${}_{2|}q_{xy}$

ii) Calculate the probability of the last death occurring before t = 3.

Solution

For the joint status we can use the usual probability formula.

$$
\begin{aligned}
{}_{2|}q_{xy} &= {}_{2}p_{xy}\, q_{x+2:y+2} \\
&= {}_{2}p_{xy} - {}_{3}p_{xy} \\
&= {}_{2}p_{x}\, {}_{2}p_{y} - {}_{3}p_{x}\, {}_{3}p_{y} \\
&= \big((0.99)(0.98)\big)\big((0.98)(0.97)\big) - \big((0.99)(0.98)(0.97)\big)\big((0.98)(0.97)(0.96)\big) \\
&= 0.922272 - (0.941094)(0.912576) = \boxed{0.063452}
\end{aligned}
$$

This is the probability of (x) dying times the probability of (y) dying.

$$
\begin{aligned}
{}_{3}q_{x}\, {}_{3}q_{y} &= (1 - {}_{3}p_{x})(1 - {}_{3}p_{y}) \\
&= (1 - 0.941094)(1 - 0.912576) = \boxed{0.0051498}
\end{aligned}
$$

Question

For two independent lives (x) and *(y):*

t	${}_tp_x$	${}_tp_{\overline{xy}}$
1	0.98	0.998
2	0.96	0.990
3	0.90	0.970
4	0.80	0.920

Determine ${}_2q_{\overline{x+2:y+2}}$

(A) $\frac{1}{60}$ (B) $\frac{1}{30}$ (C) $\frac{1}{25}$ (D) $\frac{1}{15}$ (E) $\frac{2}{25}$

Solution

${}_tp_{\overline{xy}} = {}_tp_x + {}_tp_y - {}_tp_x\,{}_tp_y$, so evaluating at $t = 2$ and $t = 4$,

$$
\begin{aligned}
0.99 &= 0.96 + {}_2p_y - 0.96\,{}_2p_y \\
0.03 &= 0.04\,{}_2p_y \\
{}_2p_y &= 0.75 \\
0.92 &= 0.80 + {}_4p_y - 0.80\,{}_4p_y \\
0.12 &= 0.2\,{}_4p_y \\
{}_4p_y &= 0.6
\end{aligned}
$$

$$
\begin{aligned}
{}_2q_{\overline{x+2:y+2}} &= {}_2q_{x+2}\,{}_2q_{y+2} \\
&= \left(\frac{0.96-0.80}{0.96}\right)\left(\frac{0.75-0.6}{0.75}\right) \\
&= \left(\frac{1}{6}\right)\left(\frac{1}{5}\right) = \boxed{\frac{1}{30}} \qquad \text{(B)}
\end{aligned}
$$

3 Present Values involving Two lives

We re-use the same assurance and annuity functions that we met when considering single lives, but using multiple life statuses instead of the single life status.

Present values of joint life and last survivor assurances

Consider an assurance under which the benefit (of 1) is paid immediately on the ending (failure) of a status u . This status u could be any joint lifetime or last survivor status, *eg: xy*, $\overline{xy}$.

Let T_u be a continuous random variable representing the future lifetime of the status u and let $f_{T_u}(t)$ be the probability density function of T_u .

How can the present value of the assurance be represented by the random variable?

3 Present Values involving Two lives

The present value of the assurance can be represented by the random variable:

$\boldsymbol{\bar{Z}_u = v^{T_u}}$ (i be the valuation rate of interest.)

The expected value of $\bar{Z}_u$ is given as:

$$\mathbf{E}[\bar{Z}_u] = \bar{A}_u = \int_{t=0}^{t=\infty} v^t f_{T_u}(t)\, dt$$

and the variance can be written as:

$$\begin{aligned}\mathrm{Var}[\bar{Z}_u] &= \mathrm{E}[\bar{Z}_u{}^2] - [E[\bar{Z}_u]]^2 \\ &= \int_{t=0}^{t=\infty} v^{2t} f_{T_u}(t)\, dt - \bar{A}_u^2 \\ &= {}^2\bar{A}_u - (\bar{A}_u)^2\end{aligned}$$

where ${}^2\bar{A}_u$ is evaluated at a valuation rate of interest of $i^* = 2i + i^2$.

Question

1. Suppose $u = xy$

i) write the mean (*ie* the expected value) and variance of the present value of an assurance payable immediately on the ending of the joint lifetime of (x) and (y).

2. Suppose u = $\overline{xy}$

i) write the mean and variance of the present value of an assurance payable immediately on the death of the last survivor of (x) or (y).

Solution

1.

Mean: $\bar{A}_{xy} = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy}\mu_{x+t:y+t}dt = \int_{t=0}^{t=\infty} v^t \, {}_tp_{xy}\left(\mu_{x+t} + \mu_{y+t}\right)dt$

Variance: ${}^2\bar{A}_{xy} - \left(\bar{A}_{xy}\right)^2$

2.

Mean: $\bar{A}_{\overline{xy}} = \int_{t=0}^{t=\infty} v^t \left({}_tp_x\mu_{x+t} + {}_tp_y\mu_{y+t} - {}_tp_{xy}\mu_{x+t:y+t}\right)dt = \bar{A}_x + \bar{A}_y - \bar{A}_{xy}$

Variance: ${}^2\bar{A}_{\overline{xy}} - \left(\bar{A}_{\overline{xy}}\right)^2 = \left({}^2\bar{A}_x + {}^2\bar{A}_y - {}^2\bar{A}_{xy}\right) - \left(\bar{A}_x + \bar{A}_y - \bar{A}_{xy}\right)^2$

3 Present Values involving Two lives

Present values of joint life and last survivor assurances [EOY]

If the assurance benefit is paid at the end of the year in which the status ends, then we can use a discrete random variable K_u with a present value function:

$$\boldsymbol{Z_u = v^{K_u + 1}}$$

In the above expression we need the extra '1' to take us to the end of the year in which the status fails, which is when the payment is made, because K_u will take us only to the start of that year.

A similar analysis for K_{xy} and $K_{\overline{xy}}$ gives the means and variances of the present values of the joint life and last survivor assurances with sums assured payable at the end of the year of death.

3 Present Values involving Two lives

Joint life

Mean: $A_{xy} = \sum_{t=0}^{t=\infty} v^{t+1} \; {}_{t|}q_{xy}$

Variance: ${}^{2}A_{xy} - (A_{xy})^2$

Last Survivor

Mean: $A_{\overline{xy}} = A_x + A_y - A_{xy}$

Variance: ${}^{2}A_{\overline{xy}} - (A_{\overline{xy}})^2 = ({}^{2}A_x + {}^{2}A_y - {}^{2}A_{xy}) - (A_x + A_y - A_{xy})^2$

3 Present Values involving Two lives

Present values of joint life and last survivor annuities

Consider an annuity under which a benefit of 1 *pa* is paid continuously so long as a status u continues.

The present value of these annuity payments can be represented by the random variable:

$\bar{a}_{\overline{T_u}|}$

The expected present value of this benefit is denoted by $\bar{a}_u$ where**:**

$$E[\bar{a}_{\overline{T_u}|}] = \bar{a}_u = \int_{t=0}^{t=\infty} \bar{a}_{\overline{t}|} f_{T_u}(t)\, dt$$

This is most simply expressed by using assurance functions:

$$E[\bar{a}_{\overline{T_u}|}] = E\left[\frac{1 - v^{T_u}}{\delta}\right] = \frac{1 - E[v^{T_u}]}{\delta} = \frac{1 - \bar{A}_u}{\delta}$$

3 Present Values involving Two lives

Present values of joint life and last survivor annuities

The variance can be expressed in a similar way:

$$\textbf{Var}(\bar{a}_{\overline{T_u}|}) = \text{var}\left(\frac{1 - v^{T_u}}{\delta}\right)$$
$$= \frac{1}{\delta^2}\,\text{var}(v^{T_u})$$
$$= \frac{1}{\delta^2}\left[{}^2\bar{A}_u - (\bar{A}_u)^2\right]$$

These results can be used to determine the means and variances for $u = xy$ (the joint life annuity) and $u = \overline{xy}$ (the last survivor annuity).

3 Present Values involving Two lives

Present values of annuities in advance and in arrears

The means and variances of the present values of annuities payable in advance and in arrears can be evaluated using the (discrete) random variables:

- **In advance:** $\ddot{a}_{\overline{K_u+1}|}$
- **In arrears:** $a_{\overline{K_u}|}$

Giving results as:

	In advance	In arrears
Mean	$\ddot{a}_u = \dfrac{1-A_u}{d}$	$a_u = \ddot{a}_u - 1 = \dfrac{(1-d)-A_u}{d}$
Variance	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$	$\dfrac{1}{d^2}\left\{{}^2A_u - (A_u)^2\right\}$

Question

Describe $\ddot{a}^{(12)}_{60:50:\,\overline{20|}}$ fully in words and calculate its value using PMA92C20 and PFA92C20 tables for the two lives respectively at 4% interest.

Solution

$\ddot{a}^{(12)}_{\overline{60:50:20}|}$ is the present value of 1 p.a. payable monthly in advance while two lives aged 60 and 50 are both still alive, for a maximum period of 20 years.

$$\ddot{a}^{(12)}_{\overline{60:50:20}|} = \ddot{a}^{(12)}_{60:50} - v^{20}\ {}_{20}p_{60:50}\,\ddot{a}^{(12)}_{80:70}$$

$$= (\ddot{a}_{60:50} - 11/24) - v^{20}\ {}_{20}p_{60:50}(\ddot{a}_{80:70} - 11/24)$$

$$= (15.161 - 0.458) - v^{20}\frac{6953.536}{9826.131}\frac{9392.621}{9952.697}(6.876 - 0.458) = 12.747$$

Question

A reversionary continuous annuity begins on the death of life x, if a second life y is then alive. Payment continues during the lifetime of y.

(a) State, using random variables, the present value of this annuity.
(b) Give an expression for the expected present value of this annuity in terms of assurance functions.

Solution

(a)

$$\bar{Z} = \begin{cases} \bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_x}|} & \text{if } T_y > T_x \\ 0 & \text{otherwise} \end{cases}$$

Alternatives

(1) $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{\min(T_x, T_y)}|}$

(2) $\bar{a}_{\overline{T_y}|} - \bar{a}_{\overline{T_{x:y}}|}$

(3) $v^{T_x}\,\bar{a}_{\overline{T_y - T_x}|}$

(b)

$$E(\bar{Z}) = \bar{a}_y - \bar{a}_{xy} = \frac{1-\bar{A}_y}{\delta} - \frac{1-\bar{A}_{xy}}{\delta} = \frac{\bar{A}_{xy}-\bar{A}_y}{\delta}$$

4 Calculations

We are now in a position to calculate numerical values for two lives. Once we have done this, we can apply the equivalence principle to calculate premiums for policies involving more than one life. In addition, we will adapt the techniques we have covered in earlier chapters for calculating reserves. This will enable us to calculate reserves for these types of policy.

Question

Using the PMA92C20 table for both lives calculate:

a) $\mu_{65:60}$

b) ${}_5p_{65:60}$

Solution

(a) $\mu_{65:60} = \mu_{65} + \mu_{60} = 0.005543 + 0.002266 = 0.007809$

(b) $_{5}p_{65:60} = \frac{l_{70}}{l_{65}} \cdot \frac{l_{65}}{l_{60}} = \frac{9238.134}{9647.797} \cdot \frac{9647.797}{9826.131} = 0.940160$

Question

(i) Calculate the expected present value of an annuity-due of 1 per annum payable annually in advance until the death of the last survivor of two lives using the following basis:

First life: male aged 70, mortality table PMA92C20
Second life: female aged 67, mortality table PFA92C20
Rate of interest: 4% per annum

Solution

(i) Expected present value:

$$\ddot{a}_{\overline{70:67}} = \ddot{a}_{70} + \ddot{a}_{67} - \ddot{a}_{70:67}$$

$$= 11.562 + 14.111 - 10.233$$

$$= 15.440$$

Question

Evaluating Premiums

William, aged 75, and Laura, aged 80, are the guardians of a child. They take out a life assurance policy that provides a payment of £25,000 immediately when the second of them dies. Level annual premiums are payable in advance whilst the policy is in force.

(i) Calculate the annual gross premium, using the basis given below.

Basis: Mortality: PMA92C20 for William, PFA92C20 for Laura
Interest: 4% pa effective
Expenses: Initial: £250
Renewal: 5% of each premium, excluding the first

Solution

(i) ***Gross annual premium***

The premium equation is:

$$P\ddot{a}_{\overline{75:80}} = 25{,}000\bar{A}_{\overline{75:80}} + 250 + 0.05P\left(\ddot{a}_{\overline{75:80}} - 1\right)$$

We can calculate $\ddot{a}_{\overline{75:80}}$ using:

$$\ddot{a}_{\overline{75:80}} = \ddot{a}_{75}^{m} + \ddot{a}_{80}^{f} - \ddot{a}_{75:80} = 9.456 + 8.989 - 6.822 = 11.623$$

The assurance factor can be calculated by premium conversion:

$$\bar{A}_{\overline{75:80}} \approx 1.04^{0.5} A_{\overline{75:80}} = 1.04^{0.5}\left(1 - d\ddot{a}_{\overline{75:80}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 11.623\right) = 0.56391$$

The premium equation becomes:

$$11.623P = 25{,}000 \times 0.56391 + 250 + 0.05P(11.623 - 1)$$

So the premium is:

$$P = \frac{14{,}347.81}{11.09185} = £1{,}293.55$$

Question

(i) Calculate $\bar{A}^{1}_{40:50}$

Basis:
Mortality μx = 0.04 throughout life for the life aged 40
μx = 0.06 throughout life for the life aged 50
Rate of interest 5% per annum

Two lives aged 40 and 50 exact purchase a policy with the benefit in part (i) above and a sum assured of 75,000. The benefit is funded by a premium payable continuously for a 30-year period or until the first death if earlier. The premium is paid at a level annual rate for the first 20 years, then reduces by 25% to be paid at the lower level annual rate for the remainder of the period.

(ii) Calculate the initial level annual premium using the basis in part (i) above.

Solution

(i) $$\bar{A}^{1}_{40:50} = \int_0^\infty v^t \, {}_tp_{40:50}\,\mu_{40+t}\,dt = .04\int_0^\infty e^{-(.04+.06+\ln 1.05)t}\,dt = .04\int_0^\infty e^{-0.14879t}\,dt$$

$$= .04\left[-\frac{e^{-.14879t}}{.14879}\right]_0^\infty = \frac{.04}{.14879} = 0.26884$$

Solution

(ii) $$\bar{a}_{40:50:\overline{20}|} = \int_0^{20} v^t\, {}_tp_{40:50}\,dt = \int_0^{20} e^{-.14879t}\,dt$$

$$= \left[-\frac{e^{-0.14879t}}{0.14879}\right]_0^{20} = \frac{1}{0.14879}(1-e^{-2.976}) = 6.378$$

$$\bar{a}_{40:50:\overline{30}|} = \int_0^{30} v^t\, {}_tp_{40:50}\,dt = \int_0^{30} e^{-.14879t}\,dt$$

$$= \left[-\frac{e^{-0.14879t}}{0.14879}\right]_0^{30} = \frac{1}{0.14879}(1-e^{-4.464}) = 6.643$$

Let Premium = P, then

$$P(0.75\times 6.643 + .25\times 6.378) = 75,000\times 0.26884$$

$$P = \frac{20163}{6.577} \Rightarrow P = 3065.7$$

4.1 Calculating Reserves

We can calculate reserves for policies based on two lives, again adapting the methods we have met earlier for reserve calculations for single life policies. The prospective calculation involves calculating the EPV of the future benefits and expenses (if appropriate), and subtracting the EPV of any future premiums.

It is important when calculating a reserve for a last survivor assurance to remember that it would be necessary to establish whether both lives remain alive or one has already died. This is because, of course, the contract still remains in force whether one or two lives remain alive. The premium being paid will still be the original premium calculated on a last survivor basis.

Question

William, aged 75, and Laura, aged 80, are the guardians of a child. They take out a life assurance policy that provides a payment of £25,000 immediately when the second of them dies. Level annual premiums are payable in advance whilst the policy is in force.

(i) Calculate the annual gross premium, using the basis given below. (Solved earlier P = 1293.55)

(ii) Calculate the gross premium prospective reserve just before the sixth premium is paid, using the basis given below, assuming that both William and Laura are still alive at that time.

Basis: Mortality: PMA92C20 for William, PFA92C20 for Laura
Interest: 4% pa effective
Expenses: Initial: £250
Renewal: 5% of each premium, excluding the first

Solution

(ii) ***Gross premium prospective reserve at time 5***

Just before the sixth premium payment (*ie* at time 5), William is aged 80 and Laura is aged 85. The gross premium prospective reserve is given by:

$$ {}_5V = 25{,}000\bar{A}_{\overline{80:85}} + 0.05P\ddot{a}_{\overline{80:85}} - P\ddot{a}_{\overline{80:85}} = 25{,}000\bar{A}_{\overline{80:85}} - 0.95P\ddot{a}_{\overline{80:85}} $$ [1]

where $P = £1{,}293.55$ from part (i).

Using the same approach as in part (i):

$$ \ddot{a}_{\overline{80:85}} = \ddot{a}^m_{80} + \ddot{a}^f_{85} - \ddot{a}_{80:85} = 7.506 + 7.220 - 5.161 = 9.565 $$ [½]

$$ \bar{A}_{\overline{80:85}} \approx 1.04^{0.5} A_{\overline{80:85}} = 1.04^{0.5}\left(1 - d\,\ddot{a}_{\overline{80:85}}\right) = 1.04^{0.5}\left(1 - \frac{0.04}{1.04} \times 9.565\right) = 0.64463 $$ [½]

So the gross premium prospective reserve at time 5 is:

$$ {}_5V = 25{,}000 \times 0.64463 - 0.95 \times 1{,}293.55 \times 9.565 = £4{,}361.68 $$ [1]

[Total 3]

Recap

We summarize all the functions and models developed for two lives:

Random Variable	Modelling
T_{xy} = $\min[T_x, T_y]$	Time to failure of the joint life status xy , ie the time until the first death of x and y
$T_{\overline{xy}}$ = $\max[T_x, T_y]$	Time to failure of the last survivor status, ie the time until the second death of x and y
K_{xy} = $\min[K_x, K_y]$	Curtate time to failure of the joint life status xy , ie the curtate time until the first death of x and y
$K_{\overline{xy}}$ = $\max[K_x, K_y]$	Curtate time to failure of the last survivor status, ie the curtate time until the second death of x and y

Recap

We have also developed the following notation, with associated formulae:

Symbol	Description	Formulae
l_{xy}	Life table survival function for two independent lives x and y	$l_x\, l_y$
μ_{xy}	Force of failure of the joint life status xy	$\mu_x + \mu_y$
tp_{xy}	Probability that the joint life status xy is still active in t years' time, ie the probability that both x and y survive for at least t years.	$tp_x\, tp_y$
$tp_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ is still active in t years' time, ie the probability that at least one of x and y survive for at least t years	$tp_x + tp_y - tp_x tp_y$
tq_{xy}	Probability that the joint life status xy fails within t years, ie the probability that at least one of x and y dies in next t years.	$tq_x + tq_y - tq_x tq_y$
$tq_{\overline{xy}}$	Probability that the last survivor status $\overline{xy}$ fails within t years, ie the probability that both x and y die in next t years.	$tq_x tq_y$

Recap

The most common benefits, and their values, are:

Function	Value (algebraic)	Value (stochastic)
$\bar{a}_{xy}$	$\int_0^\infty v^t \ {}_tp_{xy}\, dt$	$E[\bar{a}_{\overline{\min(T_x,T_y)}\rvert}]$
$\bar{a}_{\overline{xy}}$	$\int_0^\infty v^t \ {}_tp_{\overline{xy}}\, dt$	$E[\bar{a}_{\overline{\max(T_x,T_y)}\rvert}]$
$\bar{A}_{xy}$	$\int_0^\infty v^t \ {}_tp_{xy}(\boldsymbol{\mu}_{x+t} + \boldsymbol{\mu}_{y+t})dt$	$E[v^{\min(T_x,T_y)}]$
$\bar{A}_{\overline{xy}}$	$\int_0^\infty v^t \left({}_tp_y \boldsymbol{\mu}_{y+t} {}_tq_x + {}_tp_x \boldsymbol{\mu}_{x+t} {}_tq_y \right)dt$	$E[v^{\max(T_x,T_y)}]$

5 Contingent and Reversionary Benefits

We will now consider the implication of specifying the order in which the two lives die, which leads to the two types of benefit:

Contingent assurances

- These are payable on the death of one life, contingent upon another life being in a specified state (alive or dead)

Reversionary annuities

- These are payable to one life following the death of another life.

5.1 Contingent Probabilities of death

We now look at events that depend on the order in which the deaths occur. We will study two events:

- the event that (x) is the first to die of two lives (x) and (y): $\mathbf{x^1y}$
- the event that (x) is the second to die of two lives (x) and (y): $\mathbf{x^2y}$

Events that depend upon the order in which the lives die are called contingent events.

5.1 Contingent Probabilities of death

We use nq_{xy}^{1} and nq_{xy}^{2} to denote the probabilities that each of these two events occurs in the next *n* years.

- **nq_{xy}^{1} = probability that (*x*) dies within *n* years, and (*y*) dies (any time) after (*x*).**

(In this probability, *y* may or may not die within the *n*-year period – *y* can die as long as *x* is already dead.)

- **nq_{xy}^{2} = probability that (*x*) dies within *n* years, and (*y*) dies before (*x*).**

This is because the superscript over the *x* is a '2', so (*x*) has to die second.

These probabilities can be evaluated by an appropriate integration of the density functions of the random variables *Tx* and *Ty* .

5.1 Contingent Probabilities of death

We can write nq^1_{xy} as:

$$\mathbf{n}q^1_{xy} = \int_{t=0}^{t=n} {}_tp_x\, \mu_{x+t}\, {}_tp_y\, dt$$

We are calculating the product of:

- the probability that (*x*) dies at exact moment *t* (${}_tp_x\ \mu_{x+t}$ dt)
- the probability that (*y*) is still living at exact moment *t* (${}_tp_y$), *ie* that (*y*) dies after time *t* and therefore after (*x*) dies

and then summing (*ie* integrating) over all possible times at which (*x*) could die over the next *n* years.

5.1 Contingent Probabilities of death

Now we look at nq^2_{xy}

This is the probability that (x) has to die within the *n* -year period. So we model (*x*) as dying at time *t*. However, since (*x*) has to be the second life to die, we also need to include the probability that (*y*) has already died by this time, *ie* the probability that (*y*) dies before time *t*. This corresponds to the event *Ty* < *Tx* ≤ *n* .

Since the time of (*x*)'s death must be between time 0 and time *n*, the integral is:

$$\mathbf{n}q^2_{xy} = \int_0^n tp_x\,\mu_{x+t}\,tq_y\,dt$$

5.1 Contingent Probabilities of death

Relation between single death, first death and second death

If we substitute $(1 - {}_tp_y)$ for ${}_tq_y$ in the solution equation, then we have:

$$
\begin{aligned}
{}_nq^2_{xy} &= \int_0^n {}_tp_x\, \mu_{x+t}\, (1 - {}_tp_y)dt \\
&= \int_0^n {}_tp_x\, \mu_{x+t}\, dt - \int_0^n {}_tp_{xy}\, \mu_{x+t}\, dt \\
&= {}_nq_x - {}_nq^1_{xy}
\end{aligned}
$$

The truth of this expression can be argued by general reasoning if it is rewritten as:

$$\mathbf{{}_nq_x = {}_nq^1_{xy} + {}_nq^2_{xy}}$$

The right-hand side is the probability that a life aged *x* dies in an *n* -year period either before or after a life aged *y* . As these are the only two possibilities for (*x*) 's death in relation to (*y*) , this is equal to the probability that (*x*) dies at some point in the *n* -year period.

5.1 Contingent Assurances

The random variables representing the present value of contingent assurances can be expressed as functions of the random variables Tx and Ty.

For example, the present value of a sum assured of 1 paid immediately on the death of (x) provided that (y) is still alive can be expressed as:

$$\bar{Z} = v^{T_x} \quad \text{if } Tx \le Ty$$
$$= 0 \quad \text{if } Tx > Ty$$

The mean of $\bar{Z}$ is:

$$E[\bar{Z}] = \bar{A}^1_{xy} = \int_0^\infty v^t \ {}_tp_x \, \mu_{x+t} \, {}_tp_y \, dt$$
$$= \int_0^\infty v^t \ {}_tp_{xy} \, \mu_{x+t} \ dt$$

The variance of $\bar{Z}$ is:

$$Var(\bar{Z}) = {}^2\bar{A}^1_{xy} - (\bar{A}^1_{xy})^2$$

where ${}^2\bar{A}^1_{xy}$ is evaluated at a valuation rate of interest $i^2 + 2i$.

5.1 Certain Results

1. $\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$

$\bar{A}_{xy}$ is the expected present value of 1, paid immediately when the first of the two lives dies. The first life to die can either be (x) or (y), so $\bar{A}_{xy}$ is equal to the sum of an assurance that makes a payment immediately if (x) is the first life to die ($\bar{A}_{\overset{1}{x}y}$) and an assurance that makes a payment immediately if (y) is the first life to die ($\bar{A}_{x\overset{1}{y}}$)

2. $\bar{A}_{x} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y}$

$\bar{A}_{x}$ is the expected present value of 1 paid immediately on the death of (x). In relation to (y), (x) must either die first or second. Since these two possibilities are mutually exclusive and exhaustive, $\bar{A}_{x}$ is equal to the sum of an assurance that makes a payment immediately if (x) is the first life to die ($\bar{A}_{\overset{1}{x}y}$) and an assurance that makes a payment immediately if (x) is the second life to die ($\bar{A}_{\overset{2}{x}y}$).

Question

Prove the below equation:

$$\overline{\mathbf{A}}^{2}_{\mathbf{xy}} + \overline{\mathbf{A}}_{\mathbf{xy}}{}^{2} = \overline{\mathbf{A}}_{\overline{\mathbf{xy}}}$$

Solution

We know that:

$$\bar{A}_x = \bar{A}^1_{x\,y} + \bar{A}^2_{x\,y}$$

Rearranging this equation gives us:

$$\bar{A}^2_{x\,y} = \bar{A}_x - \bar{A}^1_{xy}$$

Similarly:

$$\bar{A}_{x\,y}^{\;\;2} = \bar{A}_y - \bar{A}_{xy}^{\;\;1}$$

So:

$$\begin{aligned}
\bar{A}^2_{x\,y} + \bar{A}_{x\,y}^{\;\;2} &= \bar{A}_x - \bar{A}^1_{xy} + \bar{A}_y - \bar{A}_{xy}^{\;\;1} \\
&= \bar{A}_x + \bar{A}_y - \left(\bar{A}^1_{xy} + \bar{A}_{xy}^{\;\;1}\right) \\
&= \bar{A}_x + \bar{A}_y - \bar{A}_{xy} \\
&= \bar{A}_{\overline{xy}}
\end{aligned}$$

where the last line above uses the 'last survivor (L) = single (S) + single (S) – joint (J)' result

5.1 Benefits payable EOD

If the benefit is payable at the end of the policy year in which the contingent event occurs, then we can show that:

$$A^{1}_{xy} = \sum_{t=0}^{t=\infty} v^{t+1}\ {}_tp_{xy}\ q^{1}_{x+t:y+t}$$

with analogous expressions for the variance to those derived for assurances payable immediately on the occurrence of the contingent event.

Such expressions are usually evaluated by using the approximate relationship:

$$A^{1}_{xy} \approx (1+i)^{-1/2}\ \bar{\mathrm{A}}^{1}_{\mathrm{xy}}$$

And similar expressions.

Question

CT5 September 2007 Q10
A policy provides a benefit of £500,000 immediately on the death of (y) if she dies after (x).

(i) Write down an expression in terms of T_x and T_y (random variables denoting the complete future lifetimes of (x) and (y) respectively) for the present value of the benefit under this policy. [2]

(ii) Write down an expression for the expected present value of the benefit in terms of an integral. [2]

(iii) Suggest, with a reason, the most appropriate term for regular premiums to be payable under this policy. [2]
[Total 6]

Solution

(i) $$g(T)=\begin{cases}500{,}000v^{T_y} & T_y > T_x \\ 0 & T_y \le T_x\end{cases}$$

(ii) $$E[g(T)] = 500{,}000\int_0^\infty v^t(1-{}_tp_x)\,{}_tp_y\mu_{y+t}\,dt$$

(iii) Lifetime of (*y*). If (y) dies first, no benefit is possible and if (*y*) dies second, SA becomes payable immediately. (*x*)'s lifetime is irrelevant in this context. Premium could be payable for joint lifetime of (*x*) and (*y*) but this is shorter than (*y*) and therefore we use (*y*)'s lifetime.

5.2 Reversionary Annuities

The simplest form of a reversionary annuity is one that begins on the death of (*x*) , if (*y*) is then alive, and continues during the lifetime of (*y*) . The life (*x*) is called the counter or failing life, and the life (*y*) is called the annuitant.

The random variable $\bar{Z}$ representing the present value of this reversionary annuity if it is payable continuously can be written as a function of the random variables *Tx* and *Ty* , where:

$$\bar{Z} = \bar{a}_{\overline{Ty}|} - \bar{a}_{\overline{Tx}|} \quad \text{if } Ty > Tx$$
$$= 0 \quad \text{if } Ty \le Tx$$

As an alternative to the above expressions for the present value of the reversionary annuity, we can write:

$$\bar{Z} = \bar{a}_{\overline{Ty}|} - \bar{a}_{\overline{\min(Tx,Ty)}|}$$

And because min(*Tx* ,*Ty*) = T_{xy} , we can conveniently write:

$$\bar{Z} = \bar{a}_{\overline{Ty}|} - \bar{a}_{\overline{T_{xy}}|}$$

5.2 Reversionary Annuities

Using similar methods to those used for contingent assurances, we can show that:

$$E[\bar{Z}] = \bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy} = \frac{\bar{A}_{xy} - \bar{A}_y}{\delta}$$

$$= \int_{t=0}^{t=\infty} v^t \bar{a}_{y+t} \, {}_tp_{xy} \, \mu_{x+t} \, dt$$

The variance can also be expressed as an integral in this way.

The actuarial notation for the EPV of this reversionary annuity is $\bar{a}_{x|y}$. The vertical bar in the subscript represents a period of deferment, as before. Here, it specifically means that the annuity is paid to (*y*) but is deferred for the lifetime of (*x*) .

5.2 Reversionary Annuities

If the annuity begins at the end of the year of death of (x) and is then paid annually in arrears during the lifetime of (y) , the random variable Z representing the present value of the payments can be written as a function of Kx and Ky , where:

$$Z = = a_{\overline{Ky}|} - a_{\overline{Kx}|} \quad \text{if } Ky > Kx$$
$$= 0 \quad \text{if } Ky \leq Kx$$

$$\mathbf{Z = a_{\overline{Ky}|} - a_{\overline{K_{xy}}|}}$$

Thus:

$$\mathbf{E[Z] = a_{x|y} = a_y - a_{xy} = \frac{A_{xy} - A_y}{d}}$$

Note: To find the values of reversionary annuities in arrears, we take the tabulated values of the annuities-due in the *Tables*, and subtract one from both. The two –1 terms cancel out, so that we can say:

$$a_{x|y} = a_y - a_{xy} = (\ddot{a}_y - 1) - (\ddot{a}_{xy} - 1) = \ddot{a}_y - \ddot{a}_{xy} = \ddot{a}_{x|y}$$

5.3 Joint Life functions dependent on term

EPV of joint life assurances and annuities which also depend upon term

Joint life assurances that are dependent on a fixed term of *n* years can be term assurances or endowment assurances. Their expected present values, if they are paid immediately on death, can be expressed as:

$$\bar{A}^{\,1}_{\overset{\frown}{xy}:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t:y+t} \, dt$$

$$\bar{A}_{xy:\overline{n|}} = \bar{A}^{\,1}_{\overset{\frown}{xy}:\overline{n|}} + \bar{A}_{xy:\overset{\;1}{\overline{n|}}}$$

where $\bar{A}_{xy:\overset{\;1}{\overline{n|}}} = {}_np_{xy} \, v^n$

The bracket used in the notation for the term assurance $\bar{A}^{\,1}_{\overset{\frown}{xy}:\overline{n|}}$ indicates that the joint life status must end within the fixed term of *n* years.

5.3 Joint Life functions dependent on term

EPV of joint life assurances and annuities which also depend upon term

The expected present value of the temporary joint life annuity payable continuously can be written as:

$$\bar{a}_{xy:\overline{n|}} = \int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, dt$$

5.3 Joint Life functions dependent on term

EPV of last survivor assurances and annuities that also depend upon term

Last survivor assurances that are dependent on a fixed term of *n* years can be term assurances or endowment assurances. Their expected present values can be expressed in terms of single and joint life functions.

The expressions for assurances payable immediately on death are:

$$\bar{A}_{\overline{xy}:\overline{n}|} = \bar{A}_{x:\overline{n}|} + \bar{A}_{y:\overline{n}|} - \bar{A}_{x\,y:\overline{n}|}$$

$$\bar{A}^{\;1}_{\overline{xy}:\overline{n}|} = \bar{A}^{1}_{x:\overline{n}|} + \bar{A}^{1}_{y:\overline{n}|} - \bar{A}^{\;1}_{\overbrace{xy}:\overline{n}|}$$

The expected present value of the temporary last survivor annuity payable continuously can be written as:

$$\bar{a}_{\overline{xy}:\overline{n}|} = \bar{a}_{x:\overline{n}|} + \bar{a}_{y:\overline{n}|} - \bar{a}_{x\,y:\overline{n}|}$$

5.4 Reversionary annuities dependent on term

Type 1 – an annuity payable after a fixed term has elapsed

A reversionary annuity in which the counter or failing status is a fixed term of *n* years is exactly equivalent to a deferred life annuity. The expected present value of an annuity that is paid continuously can be written:

$$\bar{a}_{\overline{n}|\,|\,y} = {}_{n|}\bar{a}_y = \bar{a}_y - \bar{a}_{y:\overline{n}|}$$

5.4 Reversionary annuities dependent on term

Type 2 – an annuity payable to (*y*) on the death of (*x*), but ceasing at time *n*

If a reversionary annuity ceases in any event after n years, *ie* is payable to (y) after the death of (x) with no payment being made after n years, the expected present value can be expressed as:

$$\bar{a}_{y:\overline{n|}} - \bar{a}_{xy:\overline{n|}}$$

We can obtain this expression using integrals, by considering the payment made at time t.

$$\int_0^n v^t \, {}_tp_y \, {}_tq_x \, dt = \int_0^n v^t \, {}_tp_y \left(1 - {}_tp_x\right) dt = \int_0^n v^t \, {}_tp_y \, dt - \int_0^n v^t \, {}_tp_{xy} \, dt = \bar{a}_{y:\overline{n|}} - \bar{a}_{xy:\overline{n|}}$$

5.4 Reversionary annuities dependent on term

Type 3 – an annuity payable to (*y*) on the death of (*x*) provided that (*x*) dies within *n* years

If the payment commences on the death of (*x*) within *n* years and then continues until the death of (*y*) , the expected present value can be expressed as:

$$\int_{t=0}^{t=n} v^t {}_tp_{xy}\, \mu_{x+t}\, \bar{a}_{y+t}\, dt = \bar{a}_y - \bar{a}_{xy} - v^n {}_np_{xy}\, (\bar{a}_{y+n} - \bar{a}_{x+n:y+n})$$

$$= \bar{a}_{x|y} - v^n\, {}_np_{xy}\, \bar{a}_{x+n|y+n}$$

This is what we might most accurately describe as a 'temporary reversionary annuity'.

Question

CT5 September 2010 Q3

Calculate the single premium payable for a temporary reversionary annuity of £12,000 per annum payable monthly in arrear to a female life currently aged 55 exact on the death of a male life currently aged 50 exact. No payment is made after 20 years from the date of purchase.

Basis:
Rate of interest 4% per annum
Mortality of male life PMA92C20
Mortality of female life PFA92C20
Expenses Nil

Solution

Value of Single Premium is:

$$12 \times 1{,}000 \times \left(a^{(12)}_{55:\overline{20|}} - a^{(12)}_{50:55:\overline{20|}} \right)$$

$$= 12{,}000\left(\left[\left(\ddot{a}_{55} - 13/24\right) - v^{20}\,{}_{20}p_{55}\left(\ddot{a}_{75} - 13/24\right)\right] - \left[\left(\ddot{a}_{50:55} - 13/24\right) - v^{20}\,{}_{20}p_{50:55}\left(\ddot{a}_{70:75} - 13/24\right)\right]\right)$$

$$= 12{,}000\left(\left[\left(18.210 - 13/24\right) - v^{20}\frac{8784.955}{9917.623}\left(10.933 - 13/24\right)\right]\right.$$

$$\left. - \left[\left(16.909 - 13/24\right) - v^{20}\frac{8784.955}{9917.623}\frac{9238.134}{9941.923}\left(8.792 - 13/24\right)\right]\right)$$

$$= 12{,}000((17.668 - 4.201) - (16.367 - 3.099))$$

$$= 2{,}388$$

5.4 Reversionary annuities dependent on term

Type 4 – an annuity payable to (*y*) on the death of (*x*) for a maximum of *n* years

If the conditions of payment say that the payment will:

- begin on the death of (*x*) and
- cease on the death of (*y*) or *n* years after the death of (*x*) (whichever event occurs first),

then the expected present value can be expressed as:

$$\int_{t=0}^{t=n} v^t \, {}_tp_{xy} \, \mu_{x+t} \, \bar{a}_{y+t:\overline{n}|} \, dt = \bar{a}_{y:\overline{n}|} + v^n \, {}_np_y \, \bar{a}_{x:y+n} - \bar{a}_{xy}$$

5.4 Reversionary annuities dependent on term

Type 5 – an annuity payable to (*y*) on the death of (*x*) and guaranteed for *n* years

The expected present value of this benefit is:

$$\bar{A}^{1}_{x:y}\,\bar{a}_{\overline{n}|} + v^n\,{}_np_y\,\bar{a}_{x\,|\,y+n}$$

The first term in the above expression is the expected present value of the guaranteed benefit, which is paid to (*y*) following the death of (*x*) .

The second term is the expected present value of the benefit paid to (*y*) once the *n*-year guarantee period has elapsed.

5.4 Reversionary annuities dependent on term

Type 6 – an annuity payable to (*y*) on the death of (*x*) and continuing for *n* years after (*y*)'s death

The expected present value of this benefit is:

$$\bar{a}_{x|y} + \bar{A}^{\ 2}_{x:y}\ \bar{a}_{\overline{n}|}$$

The first term is the expected present value of the benefit payable after the death of (*x*) while (*y*) is still alive.

The second term is the expected present value of the annuity paid for *n* years following the death of (*y*) , provided that (*y*) dies after *x* . We can think of this as an assurance that provides a lump sum payment of $\bar{a}_{\overline{n}|}$ immediately on the death of (*y*), provided (*y*) dies second

Question

CT5 September 2006 Q4
A life insurance company issues a reversionary annuity contract. Under the contract an annuity of £20,000 per annum is payable monthly for life, to a female life now aged 60 exact, on the death of a male life now aged 65 exact. Annuity payments are always on monthly anniversaries of the date of issue of the contract.

Premiums are to be paid monthly until the annuity commences or the risk ceases.

Calculate the monthly premium required for the contract.

Basis: Mortality: PFA92C20 for the female
PMA92C20 for the male
Interest: 4% per annum
Expenses: 5% of each premium payment
1.5% of each annuity payment

Solution

EPV of benefits:

$$20{,}000 a_{65|60}^{(12)} = 20{,}000(a_{60}^{(12)} - a_{65:60}^{(12)}) = 20{,}000(a_{60} - a_{65:60}) = 20{,}000(15.652 - 11.682)$$
$$= 79{,}400$$

EPV premiums:

(The premium term will be the joint lifetime of the two lives because if his death is first the annuity commences or if her death is first, there will never be any annuity.)

Let P be the monthly premium

$$12P\ddot{a}_{65:60}^{(12)} = 12P(\ddot{a}_{65:60} - \tfrac{11}{24}) = 12P(12.682 - 0.458) = 146.688P$$

Equation of value allowing for expenses:

$$1.015(79{,}400) = (1 - 0.05)(146.688P) \Rightarrow 80{,}591 = 139.3536P \Rightarrow P$$
$$= 578.32 \text{ per month}$$

6 Annuities payable m times a year

We use the following approximations:

$$\ddot{a}_x^{(m)} = \ddot{a}_x - \frac{m-1}{2m}$$

And

$$a_x^{(m)} = a_x + \frac{m-1}{2m}$$

It is important to note that the nature of the above approximation means that the single life status *x* can equally be replaced by any life status, '*u*' say.

6 Annuities payable m times a year

For joint life functions:

$$a_{xy}^{(m)} = a_{xy} + \frac{m-1}{2m}$$

For last survivor functions

$$a_{\overline{xy}}^{(m)} = a_{x}^{(m)} + a_{y}^{(m)} - a_{xy}^{(m)} = a_x + a_y - a_{xy} + \frac{m-1}{2m}$$

For reversionary annuities

$$a_{x|y}^{(m)} = a_{y}^{(m)} - a_{xy}^{(m)} = a_y - a_{xy}$$

(Notice that there is no 'correction term' in this case because they are cancelled out.)