Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: TY BSc

Subject : Pricing and Reserving for Life Insurance Products

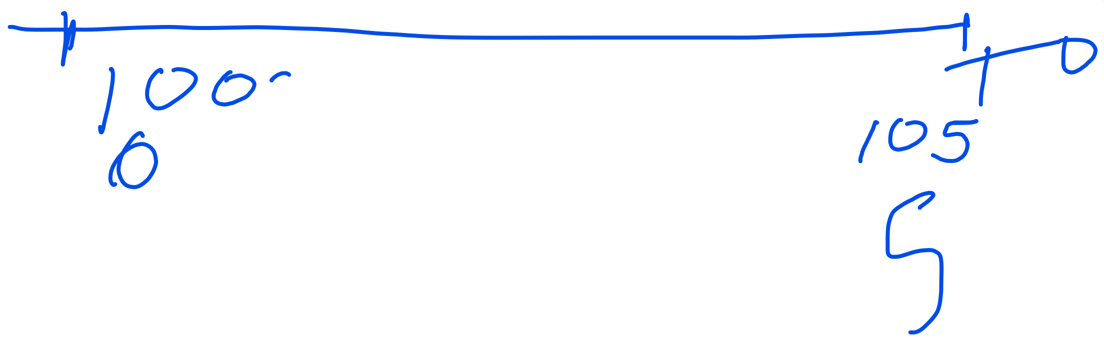
Subject Code: PUSASQF506A

Chapter: Unit 0 Chapter B

Chapter Name: The basic deterministic model

Today's Agenda

1. Interest Rates
2. Present Value of Cash flows
3. Valuing a Cash flow
4. Annuities
5. Equation of Value
6. Loan Schedules



$$i = \frac{105 - 100}{100} = 5\%$$

1.1 Effective Rate of Interest

The effective rate of interest i is the amount of money that invested at the beginning of a period will earn during the period, where interest is paid at the end of the period.

Define an **Accumulating factor A(t)** which gives the accumulated value at time t ≥ 0 of an original investment of 1.

Let i_n, be the effective rate of interest during the nth period from the date of investment. Then in terms of the accumulation factor we have;

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} \quad \text{for } n = 1,2,3,...$$

1.2 Simple and Compound Interest

What is Simple interest?

What is compound interest?

What is the difference between the two?

SI CI

0 1 2

100 110 120

$E_{SI} = \frac{110-100}{100} = 10\%$

$E_{01} = 10\%$

$E_{12} = \frac{120-110}{110} = 9.09\%$

$E_{12} = 10\%$

$E_{23} = 8.33\%$

$E_{23} = 10\%$

$C = C(1+i)^t$

$S = C(1+ti)$

0 1 2 3

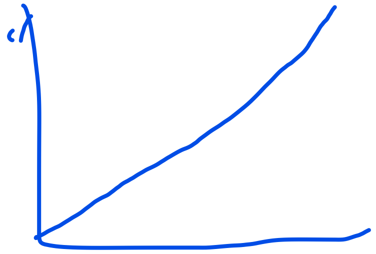
100 110 121 133.1

1.2 Simple and Compound Interest

Simple interest is interest calculated on the principal portion of a loan, investment or the original contribution to a savings account.

If C is the initial amount invested, then in general, we have a linear function for the **Accumulated value under simple interest** as:

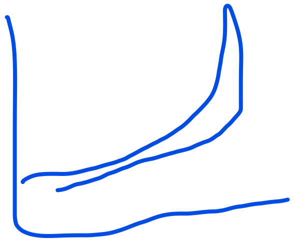
AV (t_1, t_2) = C x A (t_1, t_2) = C x $(1 + (t_2 - t_1) \times i)$
for $t_1 < t_2$



The word "compound" refers to the process of interest being reinvested to earn additional interest. With compound interest, the total investment of principal and interest earned to date is kept invested at all times

If C is the initial amount invested, then in general the **Accumulated value under compound interest** is given by the formula:

AV (t_1, t_2) = C x A (t_1, t_2) = C x $(1+i)^{(t_2 - t_1)}$
for $t_1 < t_2$



1.3 Discounting Factors

In the same way that the accumulation factor A (t_1, t_2) gives the accumulation for time t_1 to t_2 of an investment of 1 at time t_1, we define V (t_1, t_2) to be the present value factor or the discounting factor for time t_1 to t_2 of a payment of 1 due at time t_2.

Discounting factor is the reciprocal of the accumulating factor.

Thus:

$$\boxed{V(t_1, t_2) = \frac{1}{A(t_1, t_2)}}$$

0 1

100 105

$100 \, A(0,1) = 105$

$100 = \dfrac{105}{A(0,1)}$

1.3 Present Values

1.3.A PV under simple interest scenarios

We have, discounting factor as V (t) = $\frac{1}{1+it}$

Therefore, for an investment with balance C at the end of t time periods, the present value is given by;

PV (t) = C x V (t) = C x $\frac{1}{1+it}$ for t > 0

$AV = C(1+ni)$

$PV = \frac{C}{1+ni}$

$PV = \frac{C}{(1+i)^n}$

$= Cv^n$

$v = \frac{1}{1+i}$

1.3 Present Values

1.3.B PV under compound interest scenarios

Define a new symbol v, such that

$$v = \frac{1}{1+i}$$

The term v is often called a ***discount factor***, since it "discounts" the value of an investment at the end of a period to its value at the beginning of the period.

For t periods, $V(t) = \frac{1}{A(t)} = \frac{1}{(1+i)^t} = v^t$

Therefore, for an investment with balance C at the end of t time periods, the present value is given by;

$$PV(t) = C \times V(t) = \boxed{C \times \frac{1}{(1+i)^t} = C.v^t}$$

1.4 Effective Rate of Discount

$0 \quad 95 \xrightarrow{5\%} 1 \quad 100$

> The effective rate of discount d is the ratio of the amount of interest (sometimes called the “amount of discount” or just “discount”) earned during the period to the amount invested at the end of the period.

Let *d* be the effective rate of discount during the nth period from the date of investment

$$d = \frac{A(n) - A(n-1)}{A(n)} \quad \text{for } n = 1,2,3,\ldots$$

$d = \frac{100-95}{100} = 5\%$

However, from the basic definition of *i* as the ratio of the amount of interest (discount) to the principal, we obtain:

$$i = \frac{d}{1-d}$$

$i = \frac{5}{95} = 5.26\%$

By simple algebra, it is possible to express *d* as a function of i:

$$d = \frac{i}{1+i}$$

Another important relationship between i, v and d is that;

$$\boxed{d = i.v}$$

$AV_{SD} = C(1-nd) \qquad AV_{CD} = C(1-d)^n$

$AV_{SI} = C(1+ni) \qquad AV_{CI} = C(1+i)^n$

1.4 Simple and Compound Discount

The **Present Value under simple discount** is given by the general formula:

PV = C. V (t_1, t_2) = C x (1 - $(t_2 - t_1)$ x d)
for $t_1 < t_2$

The **Present Value under simple discount** is given by the general formula:

PV = C. V (t_1, t_2) = C x $(1 - d)^{(t_2 - t_1)}$
for $t_1 < t_2$

$t_1 \; t_2$

t

$$\delta \quad \dot{x} = \frac{A_n - A_{n-1}}{A_{n-1}} = \frac{A'(t)}{A(t)}$$

1.5 Force of Interest

The measures of interest defined in the preceding sections are useful for measuring interest over specified intervals of time.
It is also important to be able to measure the intensity with which interest is operating at each moment of time, i.e. over infinitesimally small intervals of time. This measure of interest at individual moments of time is called the **force of interest.**

The notation for the force of interest at time is $\boldsymbol{\delta_t}$. Thus we have the formula as:

$$\boldsymbol{\delta_t} = \frac{A\prime(t)}{A(t)}$$

Thus, $\boldsymbol{\delta_t} = \frac{d}{dt} \ln A(t)$

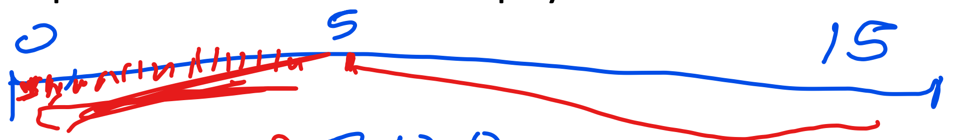
Finally, we have:

$$e^{\int_0^t \boldsymbol{\delta_r\, dr}} = \frac{A(t)}{A(0)}$$

2 Present Value of Cash flows

2.1 Discrete cash flows

If payments of Ct_1 , Ct_2, ….., Ct_n are due at times t_1 , t_2, …, t_n, and $v(t_j)$ is the discounting function, how do you calculate the present value of these payments?



100 $F(t)$ 200 400

$= 100 + 200v^5 + 400v^{15}$

$\int_0^5 F(t)\,\delta t$

$\int_{t_1}^{t_2} M(t)\,\delta t$

2 Present Value of Cash flows

2.1 Discrete cash flows

Discrete cash flow means a series of cash payments made or received at discrete points in time.

The present value of the sums Ct_1 , Ct_2, ..., Ct_n due at times t_1 , t_2, ..., t_n (where 0 < t1 < t2 < .....< tn) is given by:

$$Ct_1 v(t_1) + Ct_2\ v(t_2) + \ldots\ldots + Ct_n\ v(t_n) = \sum_{j=1}^{n} C_{t_j}\, v(t_j)$$

2 Present Value of Cash flows

2.2 Continuous cash flows

Suppose that T > 0 and that between times 0 and T an investor will be paid money continuously, the rate of payment at time t being £$\boldsymbol{\rho}$ per unit time.

If M(t) denotes the total payment made between time 0 and time t, then, by definition,

$\boldsymbol{\rho}(t) = M'(t)$ for all t

Then, if $0 \leq a < b \leq T$, the total payment received between time a and time b is

$$M(b) - M(a) = \int_a^b M'(t)\,dt$$
$$= \int_a^b \rho(t)\,dt$$

The present value of the entire cash flow is then obtained by integration as

$$\int_0^T v(t)\ \rho(t)\,dt$$

0 a b T

$\int_a^b M'(t)\,dt$

$\int_a^b \rho t\,dt\ v(t)$

3 Valuing a Cash flow

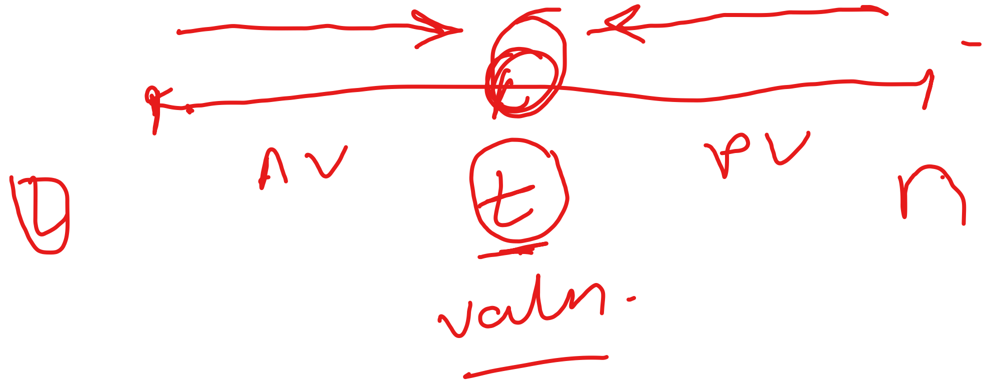
The value of payments that are due after the time of valuation is called a *discounted* value.

The value of payments that are due before the time of valuation is called an *accumulated* value.

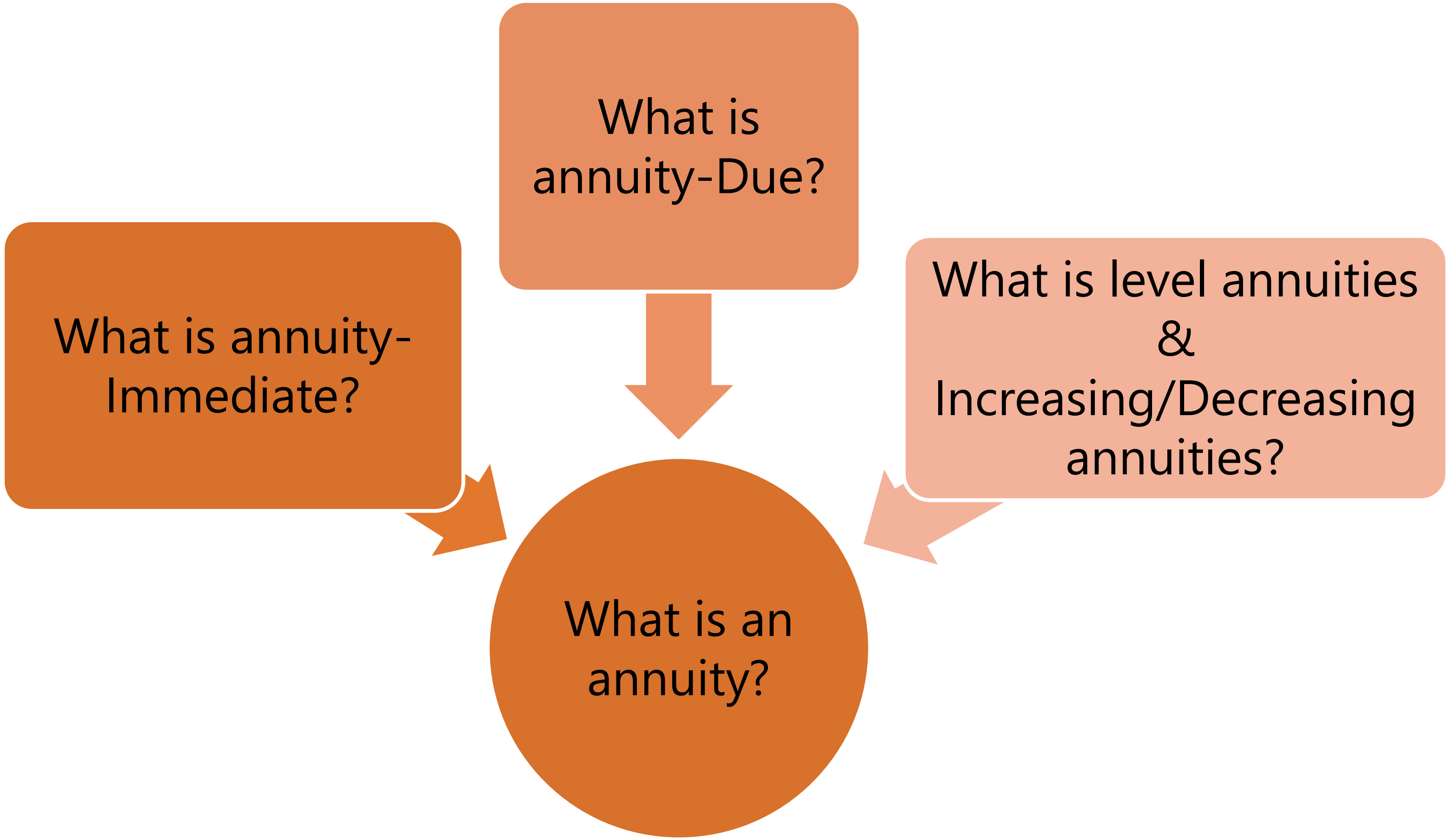
The value of a cash flow at one particular time can easily be found from the value of the cash flow at a different time.

The formula for moving along the timeline is:

[value at time t1 of the cash flow] = [value at time t2 of the cash flow] x $\left[\frac{v(t_2)}{v(t_1)}\right]$



4 Annuities


What is annuity-Due?
What is annuity-Immediate?
What is level annuities & Increasing/Decreasing annuities?
What is an annuity?

4 Annuities

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The present value of this annuity at time 0 is given by $a_{\overline{n}|}$. $= \dfrac{1 - v^n}{i}$

How do you calculate $a_{\overline{n}|}$?

0 1 2 3 . . . . . $n-1$ n

$a_{\overline{n}|}$ 1 1 1 1 2

$\ddot{a}_{\overline{n}|}$ 1 1 . . . . . .

$s_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \dots 1$

4.1 Annuities - Present Values

Annuity Arrear

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The total present value $a_{\overline{n|}}$ must equal the sum of the present values of each payment, i.e.
$a_{\overline{n|}}$ = v + v^2 + ….. + v^n

Solving this we finally get,

$$a_{\overline{n|}} = \frac{1 - v^n}{i}$$

Annuity Due

Consider a series of *n* payments made at the start of each time period. The first payment is made at time zero and the last at time *n*-1

The total present value $\ddot{a}_{\overline{n|}}$ must equal the sum of the present values of each payment, i.e.
$\ddot{a}_{\overline{n|}}$ = 1 + v + v^2 + ….. + v^{n-1}

Solving this we finally get,

$$\ddot{a}_{\overline{n|}} = \frac{1 - v^n}{d}$$

4 Annuities

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The accumulated value of this annuity at time n is given by $S_{\overline{n|}}$.

How do you calculate $S_{\overline{n|}}$?

4.2 Annuities - Accumulated Values

Annuity Arrear

Consider an annuity under which payments of 1 are made at the end of each period for n periods, where n is a positive integer.

The total accumulated value $S_{\overline{n}|}$ must equal the sum of the accumulated values of each payment, i.e.
$S_{\overline{n}|} = (1+i)^{n-1} + (1+i)^{n-2} + \ldots + 1$

Solving this we finally get,

$$S_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

Annuity Due

Consider a series of n payments made at the start of each time period. The first payment is made at time zero and the last at time n-1

The total accumulated value $\ddot{S}_{\overline{n}|}$ must equal the sum of the accumulated values of each payment, i.e.
$\ddot{S}_{\overline{n}|} = (1+i)^n + (1+i)^{n-1} + \ldots + (1+i)$

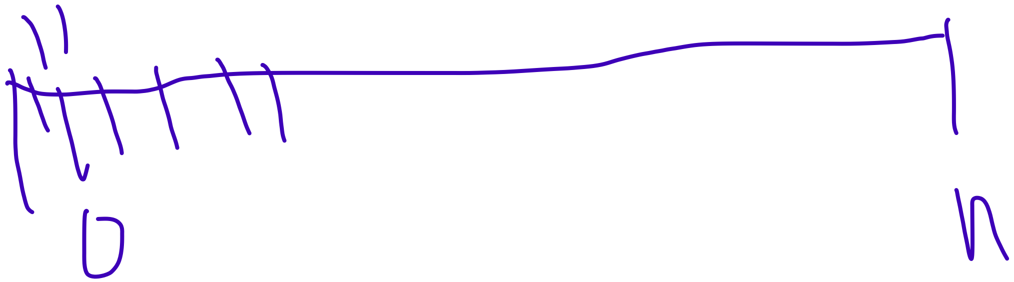
Solving this we finally get,

$$\ddot{S}_{\overline{n}|} = \frac{(1+i)^n - 1}{d}$$

4.3 Continuously Payable Annuity

The present value at time 0 of an annuity payable continuously between time 0 and time *n*, where the rate of payment per unit time is constant and equal to 1, is denoted by $\bar{a}_{\overline{n}|}$.

$$\bar{a}_{\overline{n}|} = \int_0^n 1.e^{-\delta t}\,dt$$



Solving this we finally get,

$$\bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta}$$

The accumulated value at time n is given by:

$$\bar{S}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta}$$

5 Equation of Value

What is an equation of value? State the equation and it's importance/use?

________ = ________

5 Equation of Value

An equation of value equates the present value of money received to the present value of money paid out:

‘PV income = PV outgo’

or in other words

‘PV inflow = PV outflow’

which is equal to

PV inflow - PV outflow = 0

6 Loan Schedules

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

What do we try to represent through loan schedules. What calculations does it involve?

Time.	Loan o/s start	Instal	Int Paid	Cap. Rep.	Loan o/s at end

6 Loan Schedules

A very common transaction involving compound interest is a loan that is repaid by regular instalments, at a fixed rate of interest, for a predetermined term.

What do we try to represent through loan schedules. What calculations does it involve?

6 Loan Schedules

Arjun has a Loan outstanding of Rs. 1000 at 5% p.a. for 3 years. Try to do the Calculations to fill the below table for Arjun:

Time start	Loan O/S at start	Installment	Interest paid	Capital Repaid	Loan O/S at end
0	1000	367.21	50	317.21	682.8
1	682.8	〃	34.14	333.07	349.73
2	349	〃	17.49	349	0

$1000 = X \, a_{\overline{3}|}$

$X = 367.215$

6 Loan Schedules

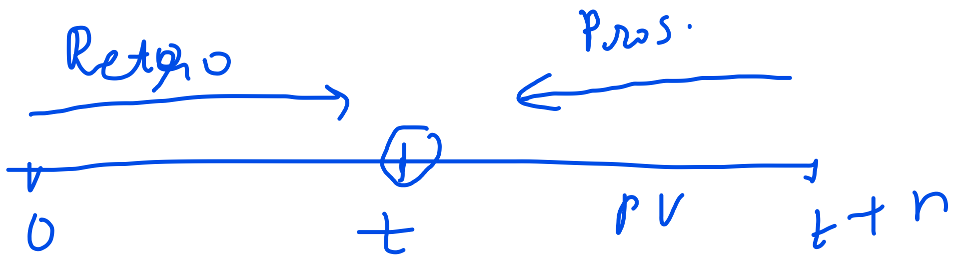
The loan payments can be expressed in the form of a table, or ‘schedule’, as follows

Year $r \rightarrow r+1$	Loan outstanding at r	Instalment at $r+1$	Interest due at $r+1$	Capital repaid at $r+1$	Loan outstanding at $r+1$
$0 \rightarrow 1$	L_0	X_1	iL_0	$X_1 - iL_0$	L_1 $= L_0 - (X_1 - iL_0)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$t \rightarrow t+1$	L_t	X_{t+1}	iL_t	$X_{t+1} - iL_t$	L_{t+1} $= L_t - (X_{t+1} - iL_t)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1 \rightarrow n$	L_{n-1}	X_n	iL_{n-1}	$X_n - iL_{n-1}$	0

6.1 Calculations

Every loan installment has two elements – interest payment and a capital repayment.

How can we calculate the capital outstanding, the capital repaid or the interest paid at any point of time during the term of the loan?



6.2 Prospective and Retrospective Methods

There are two ways to calculate the capital outstanding immediately after a repayment has been made:

Prospective method

- The loan outstanding at time t is the present (or discounted) value at time t of the future repayment instalments.
- This method involves looking forward and calculating present values of future cash flows
- L_t = X (v + v^2 + ... + v^{n-t})

  Where X is level regular instalment

Retrospective method

- The loan outstanding at time t is the accumulated value at time t of the original loan less the accumulated value at time t of the repayments to date.
- This method involves looking backward and calculating accumulated values of past cash flows
- $L_t = L_0 \, (1+i)^t$ - X $S_{\bar{t}|}$

  Where X is level regular instalment.

  L_0 is the loan outstanding at the start of the term.

6.3 Calculations – Capital and Interest Element

- We can calculate the interest element contained in a single payment by calculating the loan outstanding immediately after the previous instalment and multiplying it by the rate of interest. The capital element is the total payment less the interest payment.

- We can calculate the capital repaid in a period where there is more than one payment by subtracting the capital outstanding at the end of the period from the capital outstanding at the start of the period. The interest paid in this period is then the total payment less the capital repaid.

- The interest and capital components in the repayments for a loan can be set out in the form of a loan schedule.