

Class: TY BSc

Subject: Pricing and Reserving for Life Insurance Products

Chapter: Unit 3

Chapter Name: Gross Premiums and Reserving - 1



Today's Agenda

- 1. Gross Premiums
- 2. The Principle of Equivalence
- 2. Calculating Premiums that satisfy Probabilities
- 2. Reserving
- 2. Equality of prospective & retrospective reserves
- 2. Recursive Relationship
- 2. Profit over the Year
- 2. Mortality Profit



1.1 The concept of expenses

The income to a life insurer comes from the payments made by policyholders, called the *premiums*. The outgo arises from benefits paid to policyholders and the insurer's expenses.

By expenses we mean the costs incurred in administering a life insurance contract and, more broadly, in managing the insurance company's business as a whole.



What do you think are the expenses of a life-insurance company?



1.1 The concept of expenses

Expenses of an insurance company include spending on:

- Staff salaries
- Underwriting expenses
- Commission payments to sales intermediaries
- Costs of infra-structure (eg office buildings, computer systems)
- Costs of managing investments
- Licenses, fees etc.
- Other running costs (eg policy administration, heating, lighting, internet use).



1.1 The concept of expenses

There are three main types of expense associated with policies – initial expenses, renewal expenses and termination expenses.

Initial expenses

- These are incurred by the insurer when a policy is issued, and when we calculate a gross premium, it is conventional to assume that the insurer incurs these expenses at exactly the same time as the first premium is payable,
- There are two major types of initial expenses – commission to agents for selling a policy and underwriting expenses

Renewal expenses

- Renewal expenses are normally incurred by the insurer each time a premium is payable, and in the case of an annuity, they are normally incurred when an annuity payment is made.
- These costs arise in a variety of ways. The processing of renewal and annuity payments involves staff time and investment expenses.

Termination expenses

- Occur when a policy expires, typically on the death of a policyholder (or annuitant) or on the maturity date of a term insurance or endowment insurance.
- Generally these expenses are small, and are largely associated with the paperwork required to finalize and pay a claim.



1.2 Gross Premiums



The gross premium is the premium required to meet all the costs under an insurance contract, and is the premium that the policyholder pays. When we talk of 'the premium' for a contract, we mean the gross premium. It is also sometimes referred to as the office Premium.

The calculation explicitly allows for expenses, the premium is called a gross premium or office premium



1.2 Gross Future Loss Random Variable



We have seen the net random future loss from a policy which is in force – where the net loss, L, is defined to be: L = present value of the future outgo – present value of the future income

When the outgo includes benefits and expenses, and the income is the gross premiums, then L is referred to as the gross future loss random variable.

L = PV of benefit outgo + PV of expenses - PV of gross premium income



1.2 Example

Suppose we can allocate expenses as:

- I initial expenses in excess of those occurring regularly each year
- *e* level annual expenses
- *f* additional expenses incurred when the contract terminates

Let G denote the amount of Gross Premium.

We are assuming here that level expenses of *e* are incurred every year *including* the first, *ie* that *I* represents the amount by which the total initial expense amount *exceeds* the subsequent regular expense amount.



Write the Gross Future Loss Random Variable for a policy issued to life aged x.



1.2 Example

Suppose we can allocate expenses as:

- I initial expenses in excess of those occurring regularly each year
- e level annual expenses
- *f* additional expenses incurred when the contract terminates

Let G denote the amount of Gross Premium.

The gross future loss random variable when a policy is issued to a life aged x is:

$$L = Sv^{T_x} + I + e\overline{a}_{\overline{Tx|}} + fv^{T_x} - G\overline{a}_{\overline{Tx|}}$$

where a gross premium of G secures a sum assured of S, the sum assured is paid immediately on death and the premium is payable continuously.

It also assumes that the renewal expenses are paid continuously at a constant rate of *e pa* until the death of the policyholder (*ie* until the contract terminates).





An insurer issues a 25-year annual premium endowment insurance with sum insured \$100 000 to a select life aged 30. The insurer incurs initial expenses of \$2000 plus 50% of the first premium, and renewal expenses of 2.5% of each subsequent premium. The death benefit is payable immediately on death.

Write down the gross future loss random variable.



Let $S = 100\,000$, x = 30, n = 25 and let P denote the annual gross premium. Then

$$L_0^g = S v^{\min(T_{[x]},n)} + 2000 + 0.475P + 0.025P \ddot{a}_{\min(K_{[x]}+1,n)}$$
$$-P \ddot{a}_{\min(K_{[x]}+1,n)}$$
$$= S v^{\min(T_{[x]},n)} + 2000 + 0.475P - 0.975P \ddot{a}_{\min(K_{[x]}+1,n)}.$$

Note that the premium related expenses, of 50% of the first premium plus 2.5% of the second and subsequent premiums are more conveniently written as 2.5% of **all** premiums, plus an additional 47.5% of the first premium.

By expressing the premium expenses this way, we can simplify the gross future loss random variable, and the subsequent premium calculation.

2 The Principle of Equivalence

Under the equivalence principle, the gross premium is set such that the expected value of the future loss is zero at the start of the contract. It means that

$$\mathbf{E}[\mathbf{L}_0] = \mathbf{0}$$

which implies that

E[PV of benefit and expenses outgo - PV of gross premium income] = 0.



Thus, under the equivalence premium principle,

EPV of benefits + EPV of expenses = EPV of gross premium income.

The gross premium for a contract, given suitable mortality, interest and distribution assumptions would be found from the equation of expected present value.





CT5 April 2015 Q9

On 1 January 1999, an insurance company issued a without profit whole life policy to a life aged 45 exact. The sum assured on the policy is £125,000 which is payable at the end of the year of death. Level premiums are payable annually in advance to age 65 or until earlier death.

The company calculated the premium on the following basis:

Mortality - AM92 Select

Interest - 6% per annum

Initial expenses - 75% of the first year's premium, incurred at outset

Renewal expenses - 5% of the second and each subsequent year's premium, incurred at the beginning of the respective policy years

Claims expense - £325 payable at the end of the year of death

(i) Show that the annual premium is approximately £1,883 using equivalence principle.



(i) Let P be the annual premium for the contract. Then:

EPV of premiums is:

$$P\ddot{a}_{[45];20|} = 11.888P$$

EPV of benefits and claim expense:

$$125,325A_{[45]} = 125,325 \times 0.15918 = 19,949.23$$

EPV of other expenses:

$$0.75P + 0.05P \left[\ddot{a}_{45} - 1 \right] = 1.2944P$$

Equation of value gives

$$11.888P = 19,949.23 + 1.2944P$$

$$\Rightarrow P = \frac{19,949.23}{10.5936} = £1,883.14$$





CT5 September 2018 Q10

A life insurance company issues whole of life assurance policies to lives aged 35 exact for a sum assured of 85,000 payable at the end of year of death. Premiums are payable annually in advance.

(a) Calculate the annual office premium for each policy using the basis below.

Basis:

Mortality - AM92 Select Interest - 6% per annum Initial commission - 75% of the annual premium Initial expenses - 350

Renewal commission - 2.5% of each annual premium excluding the first Renewal expenses - 85 per annum at the start of the second and subsequent policy years.



(a) Let P be the annual premium for the policy. Then (functions at 6%):-

EPV of premiums:

$$P\ddot{a}_{[35]} = 15.993P$$

EPV of benefits:

$$85,000A_{[35]}$$

EPV of expenses:

$$0.75P + 350 + (85 + 0.025P)a_{[35]}$$

Equation of value gives:-

$$P\ddot{a}_{[35]} = 85,000 A_{[35]} + 350 + 0.75P + (0.025P + 85) a_{[35]}$$

$$P \times 15.993 = 85,000 \times 0.09475 + 350 + 0.75P + (0.025P + 85) \times 14.993$$

$$\Rightarrow P = \frac{9,678.155}{14.868175} = 650.93$$

3 Calculating premiums that satisfy probabilities

Premiums (and reserves) can be calculated which satisfy probabilities involving the gross future loss random variable.

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming 3% per annum interest, AM92 Ultimate mortality and expenses of 4% of every premium, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

If the annual premium is G, the future loss (random variable) of the policy at outset is:

$$L_0 = 10000 \ v^{K_{50}+1} - 0.96 \ \text{G} \ \ddot{a}_{\overline{K_{50}+1}}$$

We need to find the smallest value of G such that:

$$Pr(L > 0) \le 0.05$$

ie such that $Pr(L \le 0) \ge 0.95$



3 Calculating premiums that satisfy probabilities

Define G_n to be the annual premium that ensures $L_0 = 0$ for $K_{50} = n$.

This means:

$$G_{\rm n} = \frac{10000 \text{ v}^{\rm n+1}}{0.96 \text{ ä}_{\overline{\rm n}+1}}$$

Now:

$$P(L_0 \le 0 | G = G_n) = P(K_{50} \ge n)$$

However, this probability needs to be at least 0.95...

We therefore find the largest value of n that satisfies this condition, and the corresponding value of Gn is then the minimum premium required

So:
$$P(K_{50} \ge n) \ge 0.95$$

This can be then solved with the help of tables. Then we can calculate the Premium.

A premium calculated in this way is sometimes called a percentile premium.





CT5 September 2018 Q10 Continued (part b)

A life insurance company issues whole of life assurance policies to lives aged 35 exact for a sum assured of 85,000 payable at the end of year of death. Premiums are payable annually in advance.

(a) Calculate the annual office premium for each policy using the basis below.

Basis:

Mortality - AM92 Select

Interest - 6% per annum

Initial commission - 75% of the annual premium

Initial expenses - 350

Renewal commission - 2.5% of each annual premium excluding the first

Renewal expenses - 85 per annum at the start of the second and subsequent policy years.

(b) Calculate the minimum office premium the company should charge in order that the probability of making a loss on any one policy would be 5% or less.



(b) Let P' be the required minimum office premium. Then the insurer's loss random variable for this policy is given by (where K and T denote the curtate and complete future lifetime of a policyholder):-

$$L = 85,000v^{K_{[35]}+1} + 350 + 0.75P' + (0.025P' + 85)a_{\overline{K_{[35]}}} - P'\bar{a}_{\overline{K_{[35]}+1}}$$
[2]

We need to find a value of t such that

$$P(L > 0) = P(T < t) = 0.05 \Rightarrow P(T \ge t) = 0.95$$
 [0.5]

Using AM92 Select, we require:-

$$\frac{l_{[35]+t}}{l_{[35]}} \ge 0.95 \Longrightarrow l_{[35]+t} \ge 0.95 l_{[35]} = 0.95 \times 9892.9151 = 9398.269$$
 [1]

As $l_{58} = 9413.8004$ and $l_{59} = 9354.004$ then t lies between 23 and 24 so $K_{[35]} = 23$.



[0.5]

We therefore need the minimum premium such that

$$L = 0 = 85,000v^{24} + 350 + 0.75P' + (0.025P' + 85)a_{\overline{23}} - P'\ddot{a}_{\overline{24}}$$

$$\Rightarrow 0 = 85,000 \times 0.24698 + 350 + 0.75P' + (0.025P' + 85) \times 12.3034 - 13.3034P'$$

$$\Rightarrow P' = \frac{22,389.089}{12.2458} = 1828.31$$

[2]



4 Reserving

4.1 Gross Premium Prospective Reserves



The prospective reserve for a life insurance contract which is in force (that is, has been written but has not yet expired through claim or reaching the end of the term) is defined to be, for a given basis:

The expected present value of the future outgo (benefits and expenses) less the expected present value of the future income (gross premiums)

This is the prospective reserve because it looks forward to the future cash flows of the contract.



Calculating Gross Premium Prospective Reserves

The expression for the gross future loss at policy duration t can be used to determine the gross premium prospective reserve at policy duration t. This is done by determining the expected value of the gross future loss random variable.

It should be noted that the calculation of gross premium prospective reserves will use mortality, interest and expense assumptions specifically chosen for this purpose. This set of assumptions, called the gross premium valuation basis, may differ from the underlying basis used to calculate the actual gross premiums which the company charges to policyholders.

The premium used in the calculation of the gross premium reserve is always the actual gross premium being charged.





CT5 September 2013

At the beginning of 2004, a life insurance company issued a number of 20-year "special" endowment assurance policies to male lives then aged 40 exact. Each policy provides a death benefit of £75,000 payable at the end of year of death and a maturity benefit of £150,000. Premiums on each policy are payable annually in advance for the term of the policy, ceasing on earlier death.

(i) Calculate the annual gross premium for each policy using the following premium basis: [4]

Mortality AM92 Select
Interest 4% per annum
Initial commission 25% of the first annual premium
Initial expenses £400
Renewal expenses £45 per annum at the start of the second and subsequent policy years

(ii) Determine the gross premium reserve (prospective) for each policy in force at the end of the eighth policy year and for each policy in force at the end of the ninth policy year, using the same basis as above. [6]



(i) Let P be the annual premium for the policy. Then (functions at 4%):

EPV of premiums:

$$P\ddot{a}_{[40]:20]} = 13.930P$$

EPV of benefits:

$$75,000A_{[40]:\overline{20}]}^{1} + 150,000v^{20}_{20}p_{[40]}$$

=67,077.0

where:

$$v^{20}_{20} p_{[40]} = 0.45639 \times \frac{9287.2164}{9854.3036} = 0.43013$$

$$A^{1}_{[40]:\overline{20}]} = A_{[40]:\overline{20}]} - v^{20}_{20} p_{[40]} = 0.46423 - 0.43013 = 0.0341$$

$$= 75,000 \times 0.0341 + 150,000 \times 0.43013$$



EPV of expenses:

$$0.25P + 400 + 45(\ddot{a}_{[40];20]} - 1) = 0.25P + 400 + 45 \times 12.93$$
$$= 0.25P + 981.85$$

Equation of value gives:

$$13.93P = 67,077.0 + 0.25P + 981.85$$

$$\Rightarrow P = \frac{68,058.85}{13.68} = 4,975.06$$

(ii) The gross prospective policy reserve at the end of the 8th policy year is given by:

$$_{8}V = 75,000A_{48:\overline{12}|}^{1} + 150,000v_{12}^{12}p_{48} + (45-P)\ddot{a}_{48:\overline{12}|}$$

where:

$$v^{12}_{12}p_{48} = 0.62460 \times 0.95220 = 0.59474$$

$$\Rightarrow$$
 $_{8}V = 75,000 \times (0.63025 - 0.59474) + 150,000 \times 0.59474 + (45 - 4975.06) \times 9.613$

$$= 44,481.58$$

The gross prospective policy reserve at the end of the 9th policy year is given by:

$$_{9}V = 75,000A_{49:\overline{11}}^{1} + 150,000v_{11}^{11}p_{49} + (45 - P)\ddot{a}_{49:\overline{11}}$$

where:

$$v^{11}_{11}p_{49} = 0.64958 \times 0.95411 = 0.61977$$

$$\Rightarrow {}_{9}V = 75,000 \times (0.65477 - 0.61977) + 150,000 \times 0.61977 + (45 - 4975.06) \times 8.976$$

$$= 51338.28$$



4 Reserving

4.2 Gross Premium Retrospective Reserves



The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

The accumulated value allowing for interest and survivorship of the premiums received to date less

the accumulated value allowing for interest and survivorship of the benefits and expenses paid to date

The retrospective reserve on a given basis tells us how much the premiums less expenses and claims have accumulated to, averaging over a large number of policies.





For the previous question, CT5 September 2013, Calculate the reserves using the retrospective approach.

5

Equality of prospective and retrospective reserves

Conditions for equality

If:

- 1. the retrospective and prospective reserves are calculated on the same basis, and
- 2. this basis is the same as the basis used to calculate the premiums used in the reserve calculation,

then the retrospective reserve will be equal to the prospective reserve

The equality for gross premium reserves can be shown in the same way as for net premiums.

Recursive relationship between reserves for annual premium contracts

If the expected cash flows (*ie* premiums, benefits and expenses) during the policy year (t, t + 1) are evaluated and allowance is made for the time value of money, we can develop a recursive relationship linking gross premium policy values in successive years.

We illustrate this using a whole life assurance secured by level annual premiums, but the method extends to all standard contracts.

Example: Whole life assurance

Gross premium policy value at duration t = tVPremium less expenses paid at t = G - eExpected claims plus expenses paid at $t + 1 = q_{x+t}$ (S + f) Gross premium policy value at duration t + 1 = t+1



Then the equation of value at time t + 1 for these cash flows is:

$$(iV + G - e)(1+i) - q_{x+t} (S + f) = (1 - q_{x+t})_{t+1}V$$

Recursive relationship between reserves for annual premium contracts

The equation of value at time t + 1 for these cash flows is:

$$(iV + G - e)(1+i) - q_{x+t} (S + f) = (1 - q_{x+t})_{t+i}V$$

Equation above will only be satisfied if all quantities are calculated on mutually consistent bases: *ie* using the same interest, expenses and mortality assumptions for the reserves, premium and experience over the year.

Dividing through by $p_{x+t} = 1 - q_{x+t}$, we obtain a formula for calculating the reserve at the end of the year from the reserve at the beginning of the year:

$$\frac{(_{i}V + \mathbf{G} - \mathbf{e})(\mathbf{1}+\mathbf{i}) - q_{x+t}(\mathbf{S}+\mathbf{f})}{p_{x+t}} = _{t+1}V$$

The equation gives a recursive relationship between policy values in successive years.

7 Profit over the Year

The recursive equation can be reformulated to represent the profit over the year, ie:



$$PRO_t = (_tV + G - e)(1+i) - q_{x+t} (S + f) - (1 - q_{x+t})_{t+1}V$$

This profit relates to the year starting at policy duration t, that is for policy year t+1.





CT5 April 2017

Under a 20-year policy issued by a life insurance company, the benefit payable on death, at the end of the year of death, is a return of premiums paid without interest.

A premium of 631 is payable annually in advance, throughout the term of the policy.

The following information is available for a policy in force at the start of the 19th year:

Reserves at the start of the year, 18V = 17,095

Reserves at the end of the year per survivor, 19V = 18,510

Probability of death during the year = 0.015

Rate of interest earned = 4.5% per annum

Determine the profit which is expected to emerge at the end of the 19th year for each policy in force at the start of that year. Ignore expenses and all decrements other than death.



The death benefit in year 19 is $631 \times 19 = 11,989$.

Profit emerging per policy in force at the start of the year is:

$$([_{18}V + P] \times 1.045) - (11,989 \times 0.015) - ([1-0.015] \times _{19}V)$$

$$=([17,095+631]\times1.045)-(11,989\times0.015)-(0.985\times18,510)=111.49$$

8 Mortality Profit on a Single Policy

8.1 Death strain at risk (DSAR)

Consider a policy issued t years ago to a life then aged x, with sum assured S payable at the end of the year of death. Also, assume that no survival benefit is due if the life survives to t+1.

Let tV be the reserve at time t. Then we define the *death strain* in the policy year t to t+1 to be the random variable, DS, say,



DS =
$$\begin{bmatrix} 0 & \text{if the life survives to t + 1} \\ (S - _{t+1}V) & \text{if the life dies in the year [t, t+1)} \end{bmatrix}$$

The maximum death strain, (S - $_{t+1}V$) is called the death strain at risk or DSAR.



8 Mortality Profit on a Single Policy

8.2 Mortality Profit



The mortality profit is defined as: Mortality profit = Expected Death Strain – Actual Death Strain

- The EDS is the amount the company expects to pay out, in addition to the year-end reserve for a policy. EDS = q_{x+t} (S - $_{t+1}V$)
- The ADS is the amount it actually pays out, in addition to the year-end reserve.

ADS =
$$\begin{bmatrix} 0 & \text{if the life survives to t + 1} \\ (S - _{t+1}V) & \text{if the life dies in the year [t, t+1)} \end{bmatrix}$$

• If it actually pays out less than it expected to pay, there will be a profit. If the actual strain is greater than the expected strain, there will be a loss



8.3 Allowing for Survival benefits

The Death Strain is now:

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DS = \begin{bmatrix} 0 & \text{if the life survives to t } + 1 \\ (S - (t+1)V + R) \end{pmatrix} if the life dies in the year [t, t+1)
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- The expected death strain is then q_{x+t} (S ($_{t+1}V$ + R))
- The actual death strain is 0 if the life survived the year and (S (t+1)V + R) if the life died during the year
- The mortality profit is EDS ADS, as before.



8.4 **Annuities**

- In the case of an annuity of R pa, payable annually in arrears, with no death benefit, the DSAR would be -(t+1)V + R. In this case each death causes a negative strain or release of reserves.
- In the case of an annuity of R pa, payable annually in advance, with no death benefit, the DSAR would be -t+1 only. The annuity payment is made by all policies in force at the start of the year, and is not affected by whether or not the policyholder survives the year.





CT5 September 2007

On 1 January 1992 a life insurance company issued a number of 20-year pure endowment policies to a group of lives aged 40 exact. In each case, the sum assured was £75,000 and premiums were payable annually in advance.

On 1 January 2006, 500 policies were still in force. During 2006, 3 policyholders died, and no policy lapsed for any other reason.

The office calculates net premiums and net premium reserves on the following basis:

Interest: 4% per annum Mortality: AM92 Select

- (i) Calculate the profit or loss from mortality for this group for the year ending 31 December 2006. [7]
- (ii) Explain why the mortality profit or loss has arisen. [2] [Total 9]



(i)
$$P\ddot{a}_{[40]:\overline{20}|} = 75,000 A_{[40]:\overline{20}|} = 75,000 v^{20}_{20} p_{[40]}$$

 $\Rightarrow P(13.930) = (75,000)(0.45639)(0.94245)$
 $\Rightarrow P = 32,259.45/13.93 = 2,315.83$

Mortality profit = Expected Death Strain – Actual Death Strain

$$DSAR = 0 - {}_{15}V = -(75,000A_{55:\overline{5}|} - P\ddot{a}_{55:\overline{5}|})$$

$$= -(75,000v^{5}{}_{5}p_{55} - P\ddot{a}_{55:\overline{5}|})$$

$$= -\{(75,000)(0.82193)(0.97169) - (2,315.83)(4.585)\} = -49,281.51$$

$$EDS = (q_{54})(500)(-49,281.51) = (0.003976)(500)(-49,281.51) = -97,971.64$$

 $ADS = (3)(-49,281.51) = -147,844.53$

Mortality Profit = -97,971.64 - (-147,844.53) = 49,872.89 profit.



(ii) We expected $500q_{54} = 1.988$ deaths. Actual deaths were 3. With pure endowments, the death strain is negative because no death claim is paid and there is a release of reserves to the company on death. In this case, more deaths than expected means this release of reserves is greater than required by the equation of equilibrium and the company therefore makes a profit.





Question - HW

CT5 April 2005

- (i) Write down in the form of symbols, and also explain in words, the expressions death strain at risk, expected death strain and actual death strain. [6]
- (ii) A life insurance company issues the following policies:
- 15-year term assurances with a sum assured of £150,000 where the death benefit is payable at the end of the year of death 15-year pure endowment assurances with a sum assured of £75,000
- 5-year single premium temporary immediate annuities with an annual benefit payable in arrear of £25,000 On 1 January 2002, the company sold 5,000 term assurance policies and 2,000 pure endowment policies to male lives aged 45 exact and 1,000 temporary immediate annuity policies to male lives aged 55 exact. For the term assurance and pure endowment policies, premiums are payable annually in advance. During the first two years, there were fifteen actual deaths
- from the term assurance policies written and five actual deaths from each of the other two types of policy written.
- (a) Calculate the death strain at risk for each type of policy during 2004.
- (b) During 2004, there were eight actual deaths from the term assurance policies written and one actual death from each of the other two types of policy written. Calculate the total mortality profit or loss to the office in the year 2004. [13] Basis:

Interest 4% per annum
Mortality AM92 Ultimate for term assurances and pure endowments
PMA92C20 for annuities
[Total 19]





A person aged 60 exact has purchased a whole life immediate annuity with premium at time 0 being INR 2,00,000.

Calculate the largest amount of level annuity in arrears that the insurer could pay if the probability of loss from the contract is not more than 10%.

Basis:

Mortality = PMA92C20

Interest = 5%

Expenses = 1% of each annuity payment