

Class: MSc

Subject : Probability & Statistics

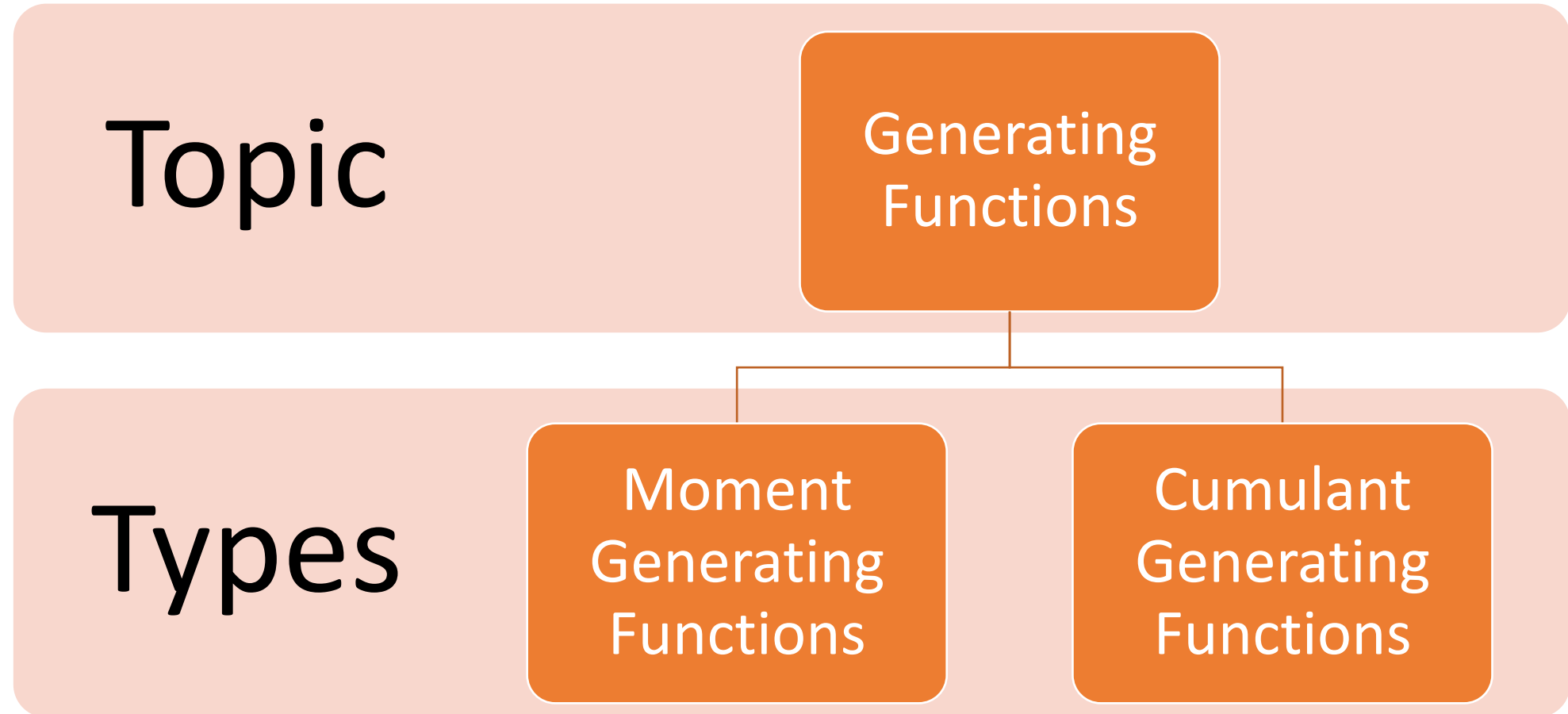
Chapter: Unit 2 Chapter 1

Chapter Name: Generating Functions

Today's Agenda

1. Introduction
2. Moment Generating Functions
3. Cumulant Generating Functions
4. Linear Functions

1 Introduction



2 Moment Generating Functions



A moment generating function (MGF) can be used to generate moments (about the origin) of the distribution of a random variable (discrete or continuous). Using MGF will simplify many calculations.

Definition: The MGF – written as $M_x(t)$, for a random variable X is given by,

$$\mathbf{M_x(t) = E[e^{(tX)}]}$$

for all values of t for which the expectations exist.

To find $M_x(t)$

If X is discrete then, $M_x(t) = \sum_x e^{(tX)} P(X = x)$

If X is continuous then, $M_x(t) = \int e^{(tX)} f(x) dx$

2.1 Finding Moments through Derivatives

The method is to differentiate the MGF with respect to t and then set $t = 0$, the r th derivative giving the r th moment about the origin.

$$M'_X(t) = E[Xe^{tX}], \text{ since } M_X(t) = E[e^{tX}] \text{ and } \frac{d}{dt}e^{tX} = Xe^{tX}.$$

$$\Rightarrow M'_X(0) = E[Xe^0] = E[X]$$

Similarly for higher orders moments.

$$M''_X(t) = E(X^2e^{tX}) \Rightarrow M''_X(0) = E(X^2)$$

$$M'''_X(t) = E(X^3e^{tX}) \Rightarrow M'''_X(0) = E(X^3)$$

The formulae for the mean and variance are:

$$\text{Mean} = M'_X(0) = E[X]$$

$$\text{Var} = M''_X(0) - [M'_X(0)]^2$$

2.2 Finding Moments through Series expansion

Expanding the exponential function and taking expected values throughout (a procedure which is justifiable for the distributions here) gives:

$$\begin{aligned} M_X(t) &= E(e^{tX}) = E\left(1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \dots\right) \\ &= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \dots \end{aligned}$$

from which it is seen that the r^{th} moment of the distribution about the origin, $E[X^r]$, is obtainable as the coefficient of $\frac{t^r}{r!}$ in the power series expansion of the MGF.

2.3 Use of MGFs

- If the distribution of a random variable X is known, in theory at least, all moments of the distribution that exist can be calculated. If the moments are specified, then the distribution can be identified.
- Without going deeply into mathematical rigour, it can in fact be said that if all moments of a random variable exist (and if they satisfy a certain convergence condition) then the sequence of moments uniquely determines the distribution of X .
- Further, if a moment generating function has been found, then there is a unique distribution with that MGF. Thus an MGF can be recognised as the MGF of a particular distribution. (There is a one-to-one correspondence between MGFs and distributions with MGFs).

2.3 Important examples – discrete distributions

The MGFs for some of the distributions introduced earlier are found as follows.

Discrete Uniform: $(e^{t/K})(1 - e^{kt})/(1 - e^t)$ for $t \neq 0$

Binomial: $(q + pe^t)^n$

Negative binomial: $[\frac{pe^t}{1-qe^t}]^k$

Poisson: $e^{\lambda(e^t-1)}$

2.3 Important examples – continuous distributions

The MGFs for some of the distributions introduced earlier are found as follows.

Continuous Uniform: $\frac{e^{bt}-e^{at}}{t(b-a)}$

Gamma: $(1 - \frac{t}{\lambda})^{-\alpha}$ for $t < \lambda$

If $t \geq \lambda$, then the power in the exponential factor in the integral is positive and therefore the answer is infinite. So, the MGF does not exist in this case.

chi-square: $(1 - 2t)^{-v/2}$

Normal: $e^{(\mu t + \frac{1}{2}\sigma^2 t^2)}$

Question

CT3 April 2005 Q3

Claim sizes in a certain insurance situation are modelled by a distribution with moment generating function $M(t)$ given by

$$M(t) = (1 - 10t)^{-2}.$$

Show that $E[X^2] = 600$ and find the value of $E[X^3]$.

Question

$$M(t) = (1 - 10t)^{-2}$$

$$M'(t) = (-2)(-10)(1-10t)^{-3} = 20(1 - 10t)^{-3}$$

$$M''(t) = (-60)(-10)(1-10t)^{-4} = 600(1-10t)^{-4} \quad \text{Putting } t = 0 \Rightarrow E[X^2] = 600$$

$$M'''(t) = (-2400)(-10)(1-10t)^{-5} = 24000(1-10t)^{-5} \quad \text{Putting } t = 0 \Rightarrow E[X^3] = 24000$$

[OR use the power series expansion $M(t) = 1 + 20t + 600t^2/2! + 24000t^3/3! + \dots$]

[OR use the result on $E[X^r]$ for a gamma(2,0.1) variable in the Yellow Book]

Question

CT3 April 2007 Q6

Consider the discrete random variable X with probability function

$$f(x) = \frac{4}{5^{x+1}}, \quad x = 0, 1, 2, \dots$$

- (i) Show that the moment generating function of the distribution of X is given by

$$M_X(t) = 4(5 - e^t)^{-1},$$

for $e^t < 5$.

- (i) Determine $E[X]$ using the moment generating function given in part (i).

Solution

$$(i) \quad M_X(t) = E[e^{tx}]$$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{4}{5} \left(\frac{1}{5}\right)^x = \frac{4}{5} \sum_{x=0}^{\infty} \left(\frac{e^t}{5}\right)^x$$

and for $e^t < 5$,

$$M_X(t) = \frac{4}{5} \frac{1}{1 - e^t/5} = 4(5 - e^t)^{-1}.$$

$$(ii) \quad M'(t) = 4e^t (5 - e^t)^{-2}$$

Mean is given by $E(X) = M'(0)$

$$\therefore E[X] = 4e^0 (5 - e^0)^{-2} = \frac{1}{4}.$$

[OR, by expansion as a power series.]

3 Cumulant Generating Functions

 The cumulant generating function, $C_X(t)$, of a random variable X is given by:

$$C_X(t) = \ln M_X(t)$$

and so $M_X(t) = e^{C_X(t)}$.

As a result, if $C_X(t)$ is known it is easy to determine $M_X(t)$.

3.1 Finding Moments

If we differentiate $C_X(t) = \ln M_X(t)$, we obtain:

$$C'_X(t) = \frac{M'_X(t)}{M_X(t)}$$

and:

$$C''_X(t) = \frac{M''_X(t)M_X(t) - (M'_X(t))^2}{M_X^2(t)}$$

Now $M_X(0) = 1$ so:

$$C'_X(0) = \frac{M'_X(0)}{M_X(0)} = \frac{E[X]}{1} = E[X]$$

and:

$$\begin{aligned} C''_X(0) &= \frac{M''_X(0)M_X(0) - (M'_X(0))^2}{M_X^2(0)} \\ &= \frac{E[X^2](1) - (E[X])^2}{1^2} \\ &= \text{var}[X] \end{aligned}$$

4 Linear Functions

Suppose X has MGF $M_X(t)$ and the distribution of a linear function $Y = a + bX$ is of interest. The MGF of Y , $M_Y(t)$ say, can be obtained from that of X as follows:

$$M_Y(t) = E[e^{tY}] = E[e^{t(a+bX)}] = e^{at} E[e^{btX}] = e^{at} M_X(bt)$$

Question

CS1 September 2021 Q7

Let X_i , $i = 1, 2, \dots, n$ be independent random variables, each following an exponential distribution with parameter b . We consider the random variable $Y = \sum_{i=1}^n X_i$.

- (i) Justify why $M_Y(t)$, the moment generating function (MGF) of variable Y , is given by

$$M_Y(t) = \left(1 - t/b\right)^{-n} \quad [2]$$

Let Z be a random variable such that the MGF of Z is $M_Z(t) = \sqrt{M_Y(t)}$.

- (ii) Determine the value of b for which Z follows a chi-square distribution, specifying the degrees of freedom of the chi-square distribution. [3]

[Total 5]

Solution

(i)

Since X_i are independent, we have that $Y = \sum_i^n X_i$ follows a gamma distribution with parameters n and b [1]

So MGF is given by $M_Y(t) = \left(1 - t/b\right)^{-n}$ [1]

(ii)

$$M_Z(t) = \sqrt{M_Y(t)} = \left(1 - t/b\right)^{-n/2} \quad [1/2]$$

The MGF of a chi-square distribution with n degrees of freedom is $(1 - 2t)^{-n/2}$ [1/2]

So $M_Z(t)$ is the MGF of a chi-square distribution with n degrees of freedom [1]

and $b = 0.5$ [1]

[Total 5]

Summary

Moment Generating Functions

$$M_X(t) = E(e^{tX}) = \sum_x e^{tx} P(X = x) \text{ or } \int_x e^{tx} f(x) dx$$

$$E(X) = M'_X(0)$$

$$\text{Var}(X) = M''_X(0) - (M'_X(0))^2$$

$$M_X(t) = 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \dots$$

Cumulant Generating Functions

$$C_X(t) = \ln M_X(t)$$

$$E(X) = C'_X(0)$$

$$\text{var}(X) = C''_X(0)$$

$$\text{skew}(X) = C'''_X(0)$$

Summary

Linear Transformations

If $Y = aX + b$ then,

$$M_Y(t) = e^{at}M_X(bt) \text{ and } C_Y(t) = at + C_X(bt)$$

The uniqueness property means that if two variables have the same MGF and CGF, then they have the same distribution.