INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Probability & Statistics

Chapter: Unit 2 Chapter 1

Chapter Name: Generating Functions

Today's Agenda

1. Introduction
2. Moment Generating Functions
3. Cumulant Generating Functions
4. Linear Functions

1 Introduction

Topic

Generating Functions

Types

- Moment Generating Functions
- Cumulant Generating Functions

2 Moment Generating Functions

A moment generating function (MGF) can be used to generate moments (about the origin) of the distribution of a random variable (discrete or continuous). Using MGF will simplify many calculations.

Definition: The MGF – written as M_x(t), for a random variable X is given by,

$\boldsymbol{M_x}$**(t) = E[**$\boldsymbol{e^{(tX)}}$**]**

for all values of t for which the expectations exist.

To find M_x(t)

If X is discrete then, $M_x(t) = \sum_x \boldsymbol{e^{(tX)}} P(X = x)$

If X is continuous then, $M_x(t) = \int \boldsymbol{e^{(tX)}} f(x)dx$

2.1 Finding Moments through Derivatives

The method is to differentiate the MGF with respect to t and then set $t = 0$, the rth derivative giving the rth moment about the origin.

$M'_X(t) = E[Xe^{tX}]$, since $M_X(t) = E[e^{tX}]$ and $\frac{d}{dt}e^{tX} = Xe^{tX}$.

$\Rightarrow M'_X(0) = E[Xe^0] = E[X]$

Similarly for higher orders moments.

$M''_X(t) = E(X^2e^{tX}) \Rightarrow M''_X(0) = E(X^2)$

$M'''_X(t) = E(X^3e^{tX}) \Rightarrow M'''_X(0) = E(X^3)$

The formulae for the mean and variance are:

Mean = $M'_X(0) = E[X]$

Var = $M''_X(0)$ - $[M'_X(0)]^2$

2.2 Finding Moments through Series expansion

Expanding the exponential function and taking expected values throughout (a procedure which is justifiable for the distributions here) gives:

$$M_X(t) \quad = E(e^{tX}) = E\left(1 + tX + \frac{t^2}{2!}X^2 + \frac{t^3}{3!}X^3 + \cdots\right)$$

$$= 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \frac{t^3}{3!}E[X^3] + \cdots$$

from which it is seen that the r^{th} moment of the distribution about the origin, $E[X^{\mathrm{r}}]$, is obtainable as the coefficient of $\frac{t^{\mathrm{r}}}{\mathrm{r}!}$ in the power series expansion of the MGF.

2.3 Use of MGFs

- If the distribution of a random variable X is known, in theory at least, all moments of the distribution that exist can be calculated. If the moments are specified, then the distribution can be identified.

- Without going deeply into mathematical rigour, it can in fact be said that if all moments of a random variable exist (and if they satisfy a certain convergence condition) then the sequence of moments uniquely determines the distribution of X.

- Further, if a moment generating function has been found, then there is a unique distribution with that MGF. Thus an MGF can be recognised as the MGF of a particular distribution. (There is a one-to-one correspondence between MGFs and distributions with MGFs).

2.3 Important examples – discrete distributions

The MGFs for some of the distributions introduced earlier are found as follows.

Discrete Uniform: $(e^{t/K})(1-e^{kt})/(1-e^{t})$ for t≠0

Binomial: $(q+pe^{t})^{\,n}$

Negative binomial: $[\frac{pe^{t}}{1-qe^{t}}]^{\,k}$

Poisson: $e^{\lambda(e^{t}-1)}$

2.3 Important examples – continuous distributions

The MGFs for some of the distributions introduced earlier are found as follows.

Continuous Uniform: $\frac{e^{bt}-e^{at}}{t(b-a)}$

Gamma: $(1-\frac{t}{\lambda})^{-\alpha}$ for t < λ

If t ≥ λ, then the power in the exponential factor in the integral is positive and therefore the answer is infinite. So, the MGF does not exist in this case.

chi-square: $(1-2t)^{-v/2}$

Normal: $e^{(\mu t+\frac{1}{2}\sigma^2 t^2)}$

Question

CT3 April 2005 Q3

Claim sizes in a certain insurance situation are modelled by a distribution with moment generating function $M(t)$ given by

$$M(t) = (1 - 10t)^{-2}.$$

Show that $E[X^2] = 600$ and find the value of $E[X^3]$.

Question

$M(t) = (1 - 10t)^{-2}$

$M'(t) = (-2)(-10)(1-10t)^{-3} = 20(1 - 10t)^{-3}$

$M''(t) = (-60)(-10)(1-10t)^{-4} = 600(1-10t)^{-4}$ Putting $t = 0 \Rightarrow E[X^2] = 600$

$M'''(t) = (-2400)(-10)(1-10t)^{-5} = 24000(1-10t)^{-5}$ Putting $t = 0 \Rightarrow E[X^3] = 24000$

[OR use the power series expansion $M(t) = 1 + 20t + 600t^2/2! + 24000t^3/3! + \ldots$]

[OR use the result on $E[X^r]$ for a gamma(2,0.1) variable in the Yellow Book]

Question

CT3 April 2007 Q6

Consider the discrete random variable X with probability function

$$f(x) = \frac{4}{5^{x+1}}, \quad x = 0, 1, 2, \ldots$$

(i) Show that the moment generating function of the distribution of X is given by

$$M_X(t) = 4(5 - e^t)^{-1},$$

for $e^t < 5$.

(ii) Determine $E[X]$ using the moment generating function given in part (i).

Solution

(i) $M_X(t) = E[e^{tx}]$

$$= \sum_{x=0}^{\infty} e^{tx} \frac{4}{5}\left(\frac{1}{5}\right)^x = \frac{4}{5}\sum_{x=0}^{\infty} \left(\frac{e^t}{5}\right)^x$$

and for $e^t < 5$,

$$M_X(t) = \frac{4}{5}\frac{1}{1-e^t/5} = 4\left(5-e^t\right)^{-1}.$$

(ii) $M'(t) = 4e^t\left(5-e^t\right)^{-2}$

Mean is given by $E(X) = M'(0)$

$$\therefore E[X] = 4e^0\left(5-e^0\right)^{-2} = \frac{1}{4}.$$

[OR, by expansion as a power series.]

3 Cumulant Generating Functions

The cumulant generating function, $C_X(t)$, of a random variable X is given by:

$\boldsymbol{C_X(t)}$ **= ln** $\boldsymbol{M_X(t)}$

and so $\boldsymbol{M_x(t) = e^{C_x(t)}}$.

As a result, if $C_X(t)$ is known it is easy to determine $M_X(t)$.

3.1 Finding Moments

If we differentiate $C_X(t) = \ln M_X(t)$, we obtain:

$$C_x'(t) = \frac{M'_x(t)}{M_x(t)}$$

and:

$$C_X''(\mathrm{t}) = \frac{M_X''(\mathrm{t})M_X(\mathrm{t}) - (M_X'(\mathrm{t}))^2}{M_X^2(\mathrm{t})}$$

Now $M_x(0) = 1$ so:

$$C_X'(0) - \frac{M_X'(0)}{M_X(0)} = \frac{E[X]}{1} = E[X]$$

and:

$$\begin{aligned} C_X''(0) &= \frac{M_X''(0)M_X(0) - (M_X'(0))^2}{M_X^2(0)} \\ &= \frac{E[X^2](1) - (E[X])^2}{1^2} \\ &= \mathrm{var}[X] \end{aligned}$$

4 Linear Functions

Suppose X has MGF $M_X(t)$ and the distribution of a linear function $Y = a + bX$ is of interest. The MGF of Y, $M_Y(t)$ say, can be obtained from that of X as follows:

$$\boldsymbol{M_Y(t) = \mathrm{E}[e^{tY}] = \mathrm{E}[e^{t(a+bX)}] = e^{at}\,\mathrm{E}[e^{btX}] = e^{at}\,M_X(bt)}$$

Question

CS1 September 2021 Q7

Let X_i, $i = 1, 2, \ldots, n$ be independent random variables, each following an exponential distribution with parameter b. We consider the random variable $Y = \sum_{i=1}^{n} X_i$.

(i) Justify why $M_Y(t)$, the moment generating function (MGF) of variable Y, is given by

$$M_Y(t) = \left(1 - {}^{t}/_{b}\right)^{-n}$$ [2]

Let Z be a random variable such that the MGF of Z is $M_z(t) = \sqrt{M_Y(t)}$.

(ii) Determine the value of b for which Z follows a chi-square distribution, specifying the degrees of freedom of the chi-square distribution. [3]

[Total 5]

Solution

(i)

Since X_i are independent, we have that $Y = \sum_i^n X_i$ follows a gamma distribution with parameters n and b [1]

So MGF is given by $M_Y(t) = \left(1 - {}^{t}/_{b}\right)^{-n}$ [1]

(ii)

$M_z(t) = \sqrt{M_Y(t)} = \left(1 - {}^{t}/_{b}\right)^{-n/2}$ [½]

The MGF of a chi-square distribution with n degrees of freedom is $(1 - 2t)^{-n/2}$ [½]

So $M_z(t)$ is the MGF of a chi-square distribution with n degrees of freedom [1]

and $b = 0.5$ [1]

[Total 5]

Summary

Moment Generating Functions

$M_X(t) = E(e^{tX}) = \sum_x e^{tx} P(X = x)$ or $\int_x e^{tx} f(x) dx$

$E(X) = M_X'(0)$

$\text{Var}\,(X) = M_X''(0) - (M_X'(0))^2$

$M_X(t) = 1 + tE(X) + \frac{t^2}{2!} E(X^2) + \frac{t^3}{3!} E(X^3) + \cdots$

Cumulant Generating Functions

$C_X(t) = \ln M_X(t)$

$E(X) = C_X'(0)$

$\text{var}(X) = C_X''(0)$

$\text{skew}(X) = C_X'''(0)$

Summary

Linear Transformations

If $Y = aX + b$ *then,*

$M_Y(t) = e^{at}M_X(bt)$ and $C_Y(t) = at + C_X(bt)$

The uniqueness property means that if two variables have the same MGF and CGF, then they have the same distribution.