INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability and Statistics 1

Chapter: Unit 3 & 4

Category: Assignment Questions

1.

An academician proposes that the joint distribution of a number of weeks of study

$$f_{XY}(x,y) = \frac{9}{10}xy^2 + \frac{1}{5}\ for\ 0 \leq x \leq 2, 0 \leq y \leq 1$$

i) Determine the Marginal distributions of X and Y.

ii) A student is selected at random, what would be the expected number of weeks of study leave and expected proportion of questions answered correctly.

iii) Compute the covariance of X and Y

2.

Let X and Y be iid random variables from an exponential distribution with mean 0.5.

i) Defining Z = Min (X ,Y), obtain the cumulative distribution function of Z.

ii) Hence, find the mean of Z.

3.

Let the random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \frac{12}{5}(x^2y + xy)\,; 0 < x, y < 1.$$

i) Find the marginal pdfs of X and Y

ii) Check for the independence of X and Y

iii) Compute E(X) and E(Y)

iv) Compute E(X/Y) and Var(X/Y)

Hence, verify E (E(X/Y)) = E(X)

4.

Suppose X has a Gamma (α, λ) distribution.

i) Derive the Moment Generating Function of X, from first principles and hence obtain its Cumulant Generating Function.

ii) Obtain an expression for the coefficient of skewness.

5.

A bivariate random variable $X = (X_1, X_2)$ has the following moment generating function.

$$M_X(t_1, t_2) = \frac{1}{3}\left(1 + e^{(t_1+2t_2)} + e^{(2t_1+t_2)}\right).$$

Determine the covariance between X_1 and X_2.

6.

Let the random variables X and Y have joint probability density function (pdf) given by

$$f(x,y) = \begin{cases} 2 & ;0 < x < y < 1 \\ 0, & otherwise. \end{cases}$$

i) Find the marginal pdf of Y.

ii) Find the conditional pdf of X given $Y = y$.

iii) Calculate the conditional mean $E(X/\ Y = 1/2\ )$.

iv) Calculate the conditional variance $V(X/\ Y = 1/2\ )$.

7.

A random variable X has a Pareto distribution with parameters α = 3 and λ = 4 and Y is a random variable such that:

$E(Y \mid X = x) = 2x + 5$ and $Var(Y \mid X = x) = x^2 + 3$

Calculate the unconditional standard deviation of Y.

8.

The random variable X has probability density

$$f(x) = \begin{cases} 8x & if\ 0 < x < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

and the random variable Y is such that the conditional density of $Y/X = x$

is $f_{Y/X}(y/X = x) = \begin{cases} \frac{1}{x} & if\ 0 < y < x \\ 0 & otherwise \end{cases}$

Find

i) The joint distribution of (X,Y)

ii) The marginal distribution of Y

iii) The mean and variance of Y, using (ii) above

9.

Show that the probability generating function of binomial distribution with mean 12 and n = 20 is $G_x(t) = (0.4 + 0.6t)^{20}$

Deduce the moment generating function of the above distribution.

10.

The number of complaints, X, handled by a call center officer in a day is modelled as Poisson with mean 5. The time (in minutes) the call center officer takes, Y, to process x complaints is modelled as having a distribution with a conditional mean and variance given by:

$E(Y|X=x) = 2x + 3$ and $V(Y|X=x) = x + 1$

Calculate the unconditional variance of the time the call center officer takes to process complaints in a day.

11.

i) Let M(t) be the Moment Generating Function (MGF) of a random variable Y. Given below are four MGFs written in terms of M(t) of four different random variables. Identify which one of the following is NOT a valid MGF.

A. $M(t) * M(3t)$

B. $e^{-3t} * M(0.5t)$

C. $\frac{2}{3} * M(t)$

D. $M(\frac{1}{3}t)$

Let X be a two-parameter exponential random variable such that X ~ Exp (λ, a). It has the following probability density function:

$$f(x) = \lambda e^{-\lambda(x-a)}, \quad x \geq a, \text{ where } \lambda, a > 0$$

ii) Show that the moment generating function of X is given by:

$$\left(1 - \frac{t}{\lambda}\right)^{-1} * e^{at}.$$

iii) Using the result derived in part (ii), calculate E(X).

iv) Which of the following is an expression representing the distribution function Fx(X) of random variable X?

A. $e^{\lambda a} * (1 - e^{-\lambda x})$
B. $e^{\lambda a} * (e^{-\lambda a} - e^{-\lambda x})$
C. $e^{-\lambda x} * (1 - e^{-\lambda a})$
D. $e^{-\lambda a} * (e^{\lambda a} - e^{-\lambda x})$

12.

The joint probability density function of random variables X and Y is:

$$f(x,y) = \begin{Bmatrix} 3e^{-(x+3y)} & , x > 0, y > 0 \\ 0 & otherwise \end{Bmatrix}$$

i) Determine $f_Y(y)$ the marginal density function of Y.

ii) Determine the conditional density function f(y | Y>4).

iii) Identify which one of the following expressions is equal to the conditional expectation E[Y | Y>4]

A. $\int_0^\infty 3e^{-3t}dt + \int_0^\infty 12e^{-3t}dt$
B. $\int_0^\infty 3e^{-3t}dt + \int_0^\infty 12te^{-3t}dt$
C. $\int_0^\infty 3te^{-3t}dt + \int_0^\infty 12e^{-3t}dt$
D. $\int_0^\infty 3te^{-3t}dt + \int_0^\infty 12te^{-3t}dt$
E. None of the above

13.

The amounts of claims of a motor insurance company are modelled as an exponential random variable with λ = 1.25 (in '000s). A data analyst is interested in assessing the probability of Y exceeding 10 (INR 10,000 if represented in absolute amounts), wherein; Y is the total of 10 independent motor claim amounts.

i) Show, using moment generating functions, that:

a) Y has gamma distribution and

b) 2.5Y has a χ^2_{20} distribution

ii) Estimate the probability of Y > 10:

A. 0.7986

B. 0.2014

C. 0.2140

D. None of the above

14.

The moment generating function (mgf) of a random variable X is given by:

$M_x(t) = \sum_{k=0}^{\infty} \frac{e^{kt-1}}{k!}$

Find $P(X = 3)$

15.

The random variable X has probability density

$$f(x) = \begin{cases} 8x & if\ 0 < x < \frac{1}{2} \\ 0 & otherwise \end{cases}$$

and the random variable Y is such that the conditional density of Y/X=x is

$$f_{Y/X}(y/X = x) = \begin{cases} \frac{1}{x} & if\ 0 < y < x \\ 0 & otherwise \end{cases}$$

Find

(i) The joint distribution of (X,Y)

ii) The marginal distribution of Y

iii) The mean and variance of Y, using (ii) above