



INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Subject: Probability and
Statistics 1

Chapter: Unit 3 & 4

Category: Assignment
Solutions

i)

$$\begin{aligned}
 f_X(x) &= \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5} \right) dy \\
 &= \left[\frac{3}{10}xy^3 + \frac{1}{5}y \right]_0^1 \\
 &= \frac{3}{10}x + \frac{1}{5}
 \end{aligned}$$

$$\begin{aligned}
 f_Y(y) &= \int_0^2 \left(\frac{9}{10}xy^2 + \frac{1}{5} \right) dx \\
 &= \left[\frac{9}{20}x^2y^2 + \frac{1}{5}x \right]_0^2 \\
 &= \frac{9}{5}y^2 + \frac{2}{5}
 \end{aligned}$$

ii)

$$\begin{aligned}
 E(X) &= \int_x x f_X(x) dx \\
 &= \int_0^2 x \left(\frac{3}{10}x + \frac{1}{5} \right) dx \\
 &= \int_0^2 \left(\frac{3}{10}x^2 + \frac{1}{5}x \right) dx \\
 &= \left[\frac{1}{10}x^3 + \frac{1}{10}x^2 \right]_0^2 \\
 &= \frac{8}{10} + \frac{4}{10} - 0 - 0 \\
 &= \frac{6}{5} = 1.2
 \end{aligned}$$

UTE OF ACTUARIAL
NTITATIVE STUDIES

$$\begin{aligned}
 E(Y) &= \int_y y f_Y(y) \, dy \\
 &= \int_0^1 y \left(\frac{9}{5} y^2 + \frac{2}{5} \right) \, dy \\
 &= \int_0^1 \left(\frac{9}{5} y^3 + \frac{2}{5} y \right) \, dy \\
 &= \left[\frac{9}{20} y^4 + \frac{1}{5} y^2 \right]_0^1 \\
 &= \frac{9}{20} + \frac{1}{5} \\
 &= \frac{13}{20} = 0.65
 \end{aligned}$$

iii)

$$\begin{aligned}
 \text{Cov}(X, Y) &= \int_x \int_y xy f_{XY}(x, y) \, dy \, dx - E(X)E(Y) \\
 &= \int_0^2 \int_0^1 xy \left(\frac{9}{10} xy^2 + \frac{1}{5} \right) \, dy \, dx - \left(\frac{6}{5} \right) \left(\frac{13}{20} \right) \\
 &= \int_0^2 \int_0^1 \left(\frac{9}{10} x^2 y^3 + \frac{1}{5} xy \right) \, dy \, dx - \frac{39}{50} \\
 &= \int_0^2 \left[\frac{9}{40} x^2 y^4 + \frac{1}{10} xy^2 \right]_0^1 \, dx - \frac{39}{50} \\
 &= \int_0^2 \left(\frac{9}{40} x^2 + \frac{1}{10} x \right) \, dx - \frac{39}{50} \\
 &= \left[\frac{3}{40} x^3 + \frac{1}{20} x^2 \right]_0^2 - \frac{39}{50} \\
 &= \frac{3}{5} + \frac{1}{5} - \frac{39}{50} \\
 &= \frac{1}{50} = 0.02
 \end{aligned}$$

TUTORIAL
STUDIES

2.

i) Given that X and Y are iid from exponential distribution with mean $\frac{1}{2}$, i.e. $\sim \text{Exp}$

$$\begin{aligned} F_Z(t) &= P(Z \leq t) = 1 - P(\text{Min}(X, Y) > t) \\ &= 1 - P(X > t \text{ and } Y > t) = 1 - P(X > t) * P(Y > t), \text{ since } X \text{ and } Y \text{ are independent} \\ &= 1 - \{1 - P(X \leq t)\} * \{1 - P(Y \leq t)\} = 1 - \{(1 - F_X(t)) * (1 - F_Y(t))\} \\ &= 1 - \{1 - (1 - e^{-2t})\} * \{1 - (1 - e^{-2t})\} \quad [\text{as } F_X(t) = F_Y(t) = 1 - e^{-2t}] \\ &= 1 - e^{-4t} \end{aligned}$$

ii) $1 - e^{-4t}$ is the CDF of an exponential distribution; hence the mean is: $\frac{1}{4}$.

3.

i) Given that $f_{X,Y}(x, y) = \frac{12}{5}(x^2y + xy)$; $0 < x, y < 1$.

$$\begin{aligned} \text{The marginal pdf of } X: h(x) &= \int_0^1 \frac{12}{5}(x^2y + xy) dy = \left[\frac{12}{5} \left(\frac{1}{2}x^2y^2 + \frac{1}{2}xy^2 \right) \right]_0^1 \\ &= \frac{12}{5} \left(\frac{1}{2}x^2 + \frac{1}{2}x \right); \quad 0 < x < 1. \end{aligned}$$

$$= \frac{6}{5}(x^2 + x); \quad 0 < x < 1$$

$$\begin{aligned} \text{The marginal pdf of } Y: g(y) &= \int_0^1 \frac{12}{5}(x^2y + xy) dx = \left[\frac{12}{5} \left(\frac{1}{3}x^3y + \frac{1}{2}x^2y \right) \right]_0^1 \\ &= \frac{12}{5} \left(\frac{1}{3}y + \frac{1}{2}y \right); \quad 0 < y < 1. \end{aligned}$$

$$= 2y; \quad 0 < y < 1$$

(ii) Clearly $f_{X,Y}(x, y) = h(x)g(y)$. The random variables are statistically independent.

$$\begin{aligned} \text{(iii)} \quad E[X] &= \int_0^1 \frac{12}{5} \left(\frac{1}{2}x^3 + \frac{1}{2}x^2 \right) dx = \left[\frac{12}{5} \left(\frac{1}{8}x^4 + \frac{1}{6}x^3 \right) \right]_{x=0}^1 \\ &= \frac{12}{5} \left(\frac{1}{8} + \frac{1}{6} \right) = 0.7 \end{aligned}$$

$$\begin{aligned} E[Y] &= \int_0^1 \frac{12}{5} \left(\frac{1}{3}y^2 + \frac{1}{2}y^2 \right) dy = \left[\frac{12}{5} \left(\frac{1}{9}y^3 + \frac{1}{6}y^3 \right) \right]_{y=0}^1 \\ &= \frac{12}{5} \left(\frac{1}{9} + \frac{1}{6} \right) = 0.67. \end{aligned}$$

$$\text{(iv)} \quad \text{We know that } E(X/Y) = \int_0^1 x f(x|y) dx = \int_0^1 x \frac{f(xy)dx}{f(y)} = \int_0^1 \frac{\frac{12}{5}(x^3y + x^2y)}{2y} dx$$

$$= (6/5) \int_0^1 (x^3 + x^2) dx = (6/5) \left[\left(\frac{1}{4}x^4 + \frac{1}{3}x^3 \right) \right]_{x=0}^1$$

$$= \frac{6}{5} \times \frac{7}{12} = \frac{7}{10}$$

$$E(E(X/Y)) = \int_0^1 \left(\frac{7}{10} \right) f(x) dx = \int_0^1 \left(\frac{7}{10} \right) f(x) dx$$

$$= \int_0^1 \left(\frac{7}{10} \right) \frac{12}{5} \left[\frac{1}{2}(x^2 + x) \right] dx$$

$$= \frac{7}{10} \times \frac{12}{5} \left[\left(\frac{1}{3}x^3 + \frac{1}{2}x^2 \right) \right]_{x=0}^1 = \frac{7}{10} \times \frac{12}{5} \times \frac{5}{6} = 0.7 = E(X)$$

$$E(X^2/Y) = \int_0^1 x^2 f(x|y) dx = \int_0^1 x^2 \frac{f(xy)dx}{f(y)} = \int_0^1 \frac{\frac{12}{5}(x^4y + x^3y)}{2y} dx$$

$$= (6/5) \int_0^1 (x^4 + x^3) dx$$

$$= (6/5) \left[\left(\frac{1}{5}x^5 + \frac{1}{4}x^4 \right) \right]_{x=0}^1 = \frac{6}{5} \times \frac{9}{20} = \frac{54}{100}.$$

$$V(X/Y) = E(X^2/Y) - (E(X/Y))^2 = \frac{54}{100} - \left(\frac{7}{10} \right)^2 = \frac{1}{20}.$$

[If the candidate has answered using $E[X/Y] = E[X]$ and $V[X/Y] = V[X]$ on computation of $E[X^2]$ and $V[X]$ full credit is to be given

4.

i) By definition, the moment generating function of $\text{Gamma}(\alpha, \lambda)$:

$$M_X(t) = E(e^{tX}) = \int_0^\infty \left(e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} \right) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(x^{\alpha-1} e^{-(\lambda-t)x} \right) dx$$

$$M_X(t) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^\infty \left(\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x} \right) dx$$

The integrand is pdf of $\text{Gamma}(\alpha, \lambda - t)$ and the value of integral is 1.

Therefore, $M_X(t) = \left(\frac{\lambda}{\lambda-t} \right)^\alpha$, provided $t < \lambda$.

Dividing the numerator and denominator by λ gives:

$$M_X(t) = \left(\frac{1}{1-\frac{t}{\lambda}} \right)^\alpha = \left(1 - \frac{t}{\lambda} \right)^{-\alpha} \quad t < \lambda$$

The cumulant generating function is:

ACTUARIAL
STUDIES

$$C_X(t) = \log M_X(t) = -\alpha \log \left(1 - \frac{t}{\lambda}\right)$$

ii) The coefficient of skewness = $\frac{Skew(X)}{Var(X)^{1.5}}$

We know that $Var(X) = C_X''(0)$ and $Skew(X) = C_X'''(0)$

$$C_X'(t) = \frac{\alpha}{\lambda} \left(1 - \frac{t}{\lambda}\right)^{-1}$$

$$C_X''(t) = \frac{\alpha}{\lambda^2} \left(1 - \frac{t}{\lambda}\right)^{-2} \Rightarrow Var(X) = C_X''(0) = \frac{\alpha}{\lambda^2}$$

$$C_X'''(t) = \frac{2\alpha}{\lambda^3} \left(1 - \frac{t}{\lambda}\right)^{-3} \Rightarrow Skew(X) = C_X'''(0) = \frac{2\alpha}{\lambda^3}$$

Hence, the coefficient of skewness = $\frac{2\alpha/\lambda^3}{(\alpha/\lambda^2)^{1.5}} = \frac{2\alpha}{\alpha^{1.5}} = \frac{2}{\sqrt{\alpha}}$

5.

The moment generating function of $X_1 = M_{X_1}(t_1) = E(e^{t_1 X_1})$

$$\begin{aligned} &= M_X(t_1, 0) = \frac{1}{3} (1 + e^{(t_1+2 \times 0)} + e^{(2t_1+0)}) \\ &= \frac{1}{3} (1 + e^{t_1} + e^{2t_1}) \end{aligned}$$

The expected value of X_1 is obtained by taking first derivative of its MGF and evaluating at $t_1=0$.

$$\text{Thus, } E(X_1) = M_{X_1}'(0) = \frac{1}{3} (1 + e^0 + e^0) = 1$$

Similarly using the mgf of X_2 , $E(X_2)$ is shown to be 1

$E(X_1 X_2)$ is computed by taking the second cross-partial derivative of joint moment generating function evaluated at $(t_1, t_2) = (0, 0)$:

$$\begin{aligned} \frac{\partial^2 M_{X_1, X_2}(t_1, t_2)}{\partial t_1 \partial t_2} &= \frac{\partial}{\partial t_1} \left(\frac{\partial}{\partial t_2} \left(\frac{1}{3} [1 + \exp(t_1 + 2t_2) + \exp(2t_1 + t_2)] \right) \right) \\ &= \frac{\partial}{\partial t_1} \left(\frac{1}{3} [2 \exp(t_1 + 2t_2) + \exp(2t_1 + t_2)] \right) \\ &= \frac{1}{3} [2 \exp(t_1 + 2t_2) + 2 \exp(2t_1 + t_2)] \end{aligned}$$

$$\text{Thus, } E(X_1 X_2) = \frac{4}{3}$$

$$\text{Cov}(X_1 X_2) = E(X_1 X_2) - E(X_1)E(X_2) = \frac{4}{3} - 1 \times 1 = \frac{1}{3} = 0.33$$

6.

i) The marginal pdf of Y:

$$i. f_1(y) = \int_0^y 2 dx = \begin{cases} 2y & 0 < y < 1 \\ 0, & \text{otherwise.} \end{cases}$$

ii) The conditional pdf of X given Y: $f(x/y)$ is

$$i. f(x/y) = \frac{f(x,y)}{f_1(y)} = \begin{cases} \frac{2}{2y}; & 0 < x < y; 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

iii) The conditional mean: $E(X/Y = y) = \int_0^y x \frac{1}{y} dx = \frac{y}{2}, 0 < y < 1$

$$1. = \frac{1}{4} = 0.25 \quad \text{when } y = \frac{1}{2}$$

iv) The conditional variance: $V(X/Y = y) = E(X^2/Y = y) - (E(X/Y = y))^2$

$$a. E(X^2/Y = y) = \int_0^y x^2 \frac{1}{y} dx = \frac{y^2}{3}; 0 < y < 1$$

$$1. = \frac{1}{12} = 0.083 \quad \text{when } y = \frac{1}{2}$$

$$\text{Hence, } V(X/Y = \frac{1}{2}) = \frac{1}{12} - \left(\frac{1}{4}\right)^2 = \frac{1}{48} = 0.0208.$$

7.

The mean and variance of the Pareto distribution are given by:

$$E[X] = \frac{\lambda}{\alpha - 1} = \frac{4}{3 - 1} = 2$$

$$Var[X] = \frac{\alpha \lambda^2}{(\alpha - 1)^2(\alpha - 2)} = \frac{3(4)^2}{(3 - 1)^2(3 - 2)} = 12$$

From page 16 of the Tables $Var(Y) = Var[E(Y|X)] + E[Var(Y|X)]$:

$$Var(Y) = Var[2X + 5] + E[X^2 + 3]$$

$$\text{Var}(Y) = 4 \text{Var}[X] + E[X^2] + 3$$

$$\text{We know that } E[X^2] = \text{Var}[X] + (E[X])^2 = 12 + 2^2 = 16$$

$$\text{Var}(Y) = 4(12) + 16 + 3 = 67$$

$$\text{Hence, the standard deviations is } \sqrt{67} = 8.1854$$

8.

(i) The Joint density of (X, Y) :

$$\text{We know that } f(y|x) = \frac{f(x, y)}{f(x)}.$$

$$\text{Hence, } f(x, y) = f(y|x)f(x)$$

$$= \frac{1}{x} (8x)$$

$$\text{Thus, } f(x, y) = \begin{cases} 8 & \text{if } 0 < x < \frac{1}{2} \text{ and } 0 < y < x \\ 0 & \text{Otherwise.} \end{cases}$$

(ii) The marginal density of Y :

$$f(y) = \int f(x, y) dx = \int_y^{0.5} 8 dx = 4(1 - 2y) \text{ for } 0 < y < \frac{1}{2}.$$

(iii) The Mean and Variance of X and Y :

$$E(Y) = \int_0^{0.5} (y) 4(1 - 2y) dy = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}.$$

$$V(Y) = E(Y^2) - (E(Y))^2.$$

$$E(Y^2) = \int_0^{0.5} y^2 4(1 - 2y) dy = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

$$V(Y) = \frac{1}{24} - \left(\frac{1}{6}\right)^2 = \frac{1}{72}.$$

9.

(i)

Given that the mean of the binomial distribution is $(np) = 12$ and $n = 20$.

Hence $p = 0.6$. That is the distribution is binomial $(20, 0.6)$

The PGF of binomial distribution is given by

$$\begin{aligned} G_X(t) &= E(t^X) = \sum_{x=0}^{20} t^x P(X=x) \\ &= \sum_{x=0}^{20} t^x \binom{20}{x} (0.6)^x (1-0.6)^{(20-x)} \\ &= \sum_{x=0}^{20} \binom{20}{x} (0.6t)^x (1-0.6)^{(20-x)} \\ &= (0.6t + (1-0.6))^{20} \\ &= (0.4 + 0.6t)^{20} \end{aligned}$$

For MGF, replace t by e^t in the above expression

$$M_X(t) = (0.4 + 0.6e^t)^{20}$$

10.

$$\begin{aligned} V(Y) &= E[V(Y|X)] + V[E(Y|X)] \\ &= E(X+1) + V(2X+3) \\ &= E(X) + 1 + 4V(X) \\ &= 5 + 1 + 4(5) \\ &= 26 \end{aligned}$$

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