INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability and Statistics 1

Chapter: Unit 3 & 4

Category: Assignment Solutions

1.

i)

$$
\begin{aligned}
f_X(x) &= \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) \mathrm{d}y \\
&= \left[\frac{3}{10}xy^3 + \frac{1}{5}y\right]_0^1 \\
&= \frac{3}{10}x + \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \int_0^2 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) \mathrm{d}x \\
&= \left[\frac{9}{20}x^2y^2 + \frac{1}{5}x\right]_0^2 \\
&= \frac{9}{5}y^2 + \frac{2}{5}
\end{aligned}
$$

ii)

$$
\begin{aligned}
E(X) &= \int_x x f_X(x)\, \mathrm{d}x \\
&= \int_0^2 x\left(\frac{3}{10}x + \frac{1}{5}\right) \mathrm{d}x \\
&= \int_0^2 \left(\frac{3}{10}x^2 + \frac{1}{5}x\right) \mathrm{d}x \\
&= \left[\frac{1}{10}x^3 + \frac{1}{10}x^2\right]_0^2 \\
&= \frac{8}{10} + \frac{4}{10} - 0 - 0 \\
&= \frac{6}{5} = 1.2
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= \int_y y f_Y(y)\,\mathrm{d}y \\
&= \int_0^1 y\left(\frac{9}{5}y^2 + \frac{2}{5}\right)\,\mathrm{d}y \\
&= \int_0^1 \left(\frac{9}{5}y^3 + \frac{2}{5}y\right)\,\mathrm{d}y \\
&= \left[\frac{9}{20}y^4 + \frac{1}{5}y^2\right]_0^1 \\
&= \frac{9}{20} + \frac{1}{5} \\
&= \frac{13}{20} = 0.65
\end{aligned}
$$

iii)

$$
\begin{aligned}
Cov(X,Y) &= \int_x \int_y xy f_{XY}(x,y)\,\mathrm{d}y\,\mathrm{d}x - E(X)E(Y) \\
&= \int_0^2 \int_0^1 xy\left(\frac{9}{10}xy^2 + \frac{1}{5}\right)\,\mathrm{d}y\,\mathrm{d}x - \left(\frac{6}{5}\right)\left(\frac{13}{20}\right) \\
&= \int_0^2 \int_0^1 \left(\frac{9}{10}x^2y^3 + \frac{1}{5}xy\right)\,\mathrm{d}y\,\mathrm{d}x - \frac{39}{50} \\
&= \int_0^2 \left[\frac{9}{40}x^2y^4 + \frac{1}{10}xy^2\right]_0^1\,\mathrm{d}x - \frac{39}{50} \\
&= \int_0^2 \left(\frac{9}{40}x^2 + \frac{1}{10}x\right)\,\mathrm{d}x - \frac{39}{50} \\
&= \left[\frac{3}{40}x^3 + \frac{1}{20}x^2\right]_0^2 - \frac{39}{50} \\
&= \frac{3}{5} + \frac{1}{5} - \frac{39}{50} \\
&= \frac{1}{50} = 0.02
\end{aligned}
$$

2.

i) Given that X and Y are *iid* from exponential distribution with mean $\frac{1}{2}$ i.e. ~ *Exp*

$$Fz(t) = P(Z \leq t) = 1 - P(Min(X,Y) > t)$$
$$= 1 - P(X > t \text{ and } Y > t) = 1 - P(X > t) * P(Y > y), \text{since } X \text{ and } Y \text{ are independent}$$
$$= 1 - \{1 - P(X \leq t)\} * \{1 - P(Y \leq t)) = 1 - \{(1 - Fx(t)) * (1 - Fy(t))\}$$
$$= 1 - \{1 - (1 - e^{-2t})\} * \{1 - (1 - e^{-2t})\} \qquad [as\ Fx(t) = Fy(t) = 1 - e^{-\lambda t}]$$
$$= 1 - e^{-4t}$$

ii) $1 - e^{-4t}$ is the CDF of an exponential distribution; hence the mean is: $\frac{1}{4}$.

3.

i) Given that $f_{X,Y}(x,y) = \frac{12}{5}(x^2y + xy)\,; 0 < x, y < 1.$

The marginal pdf of X: $h(x) = \int_0^1 \frac{12}{5}(x^2y + xy)\, dy = \left[\frac{12}{5}\left(\frac{1}{2}x^2y^2 + \frac{1}{2}xy^2\right)\right]_0^1$

$$= \frac{12}{5}\left(\frac{1}{2}x^2 + \frac{1}{2}x\right) \;;\quad 0 < x < 1.$$

$$= \frac{6}{5}\,(x^2 + x);\;\; 0 < x < 1$$

The marginal pdf of Y: $g(y)$= $\int_0^1 \frac{12}{5}(x^2y + xy)\, dx$ =$\left[\frac{12}{5}\left(\frac{1}{3}x^3y + \frac{1}{2}x^2y\right)\right]_0^1$

$$=\frac{12}{5}\left(\frac{1}{3}y + \frac{1}{2}y\right);\quad 0 < y < 1.$$

$$= 2y \;;\quad 0 < y < 1$$

(ii) Clearly $f_{X,Y}(x,y) = h(x)g(y)$. The random variables are statistically independent.

(iii) $E[X]$ $= \int_0^1 \frac{12}{5}\left(\frac{1}{2}x^3 + \frac{1}{2}x^2\right) dx = \left[\frac{12}{5}\left(\frac{1}{8}x^4 + \frac{1}{6}x^3\right)\right]_{x=0}^{1}$

$$= \frac{12}{5}\left(\frac{1}{8} + \frac{1}{6}\right) = 0.7$$

$E\,[Y]$ $= \int_0^1 \frac{12}{5}\left(\frac{1}{3}y^2 + \frac{1}{2}y^2\right) dy = \left[\frac{12}{5}\left(\frac{1}{9}y^3 + \frac{1}{6}y^3\right)\right]_{y=0}^{1}$

$$= \frac{12}{5}\left(\frac{1}{9} + \frac{1}{6}\right) = 0.67.$$

(iv) We know that $E(X/Y) = \int_0^1 x\, f(x|y)dx$ = $\int_0^1 x\, \frac{f(xy)dx}{f(y)}$ = $\int_0^1 \frac{\frac{12}{5}(x^3y + x^2y)}{2y} dx$

$$= (6/5)\int_0^1 (x^3 + x^2)\, dx = (6/5)\left[\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\right]_{x=0}^{1}$$

$$= \frac{6}{5} \times \frac{7}{12} = \frac{7}{10}$$

$$E(E(X/Y)) = \int_0^1 \left(\frac{7}{10}\right) f(x)dx = \int_0^1 \left(\frac{7}{10}\right) f(x)dx$$

$$= \int_0^1 \left(\frac{7}{10}\right)\frac{12}{5}\left[\frac{1}{2}(x^2 + x)\right]dx$$

$$= \frac{7}{10} \times \frac{12}{10}\left[\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\right]_{x=0}^{1} = \frac{7}{10} \times \frac{12}{10} \times \frac{5}{6} = 0.7 = E(X)$$

$$E(X^2/Y) = \int_0^1 x^2 f(x|y)dx = \int_0^1 x^2 \frac{f(xy)dx}{f(y)} = \int_0^1 \frac{\frac{12}{5}(x^4y + x^3y)}{2y} dx$$

$$= (6/5)\int_0^1 (x^4 + x^3)\, dx$$

$$= (6/5)\left[\left(\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\right]_{x=0}^{1} = \frac{6}{5} \times \frac{9}{20} = \frac{54}{100}.$$

$$V(X/Y) = E(X^2/Y) - (E(X/Y))^2 = \frac{54}{100} - \left(\frac{7}{10}\right)^2 = \frac{1}{20}.$$

[If the candidate has answered using $E[X/Y] = E[X]$ and $V[X/Y] = V[X]$ on computation of $E[X^2]$ and $V[X]$ full credit is to be given

4.

i) By definition, the moment generating function of Gamma(α, λ):

$$M_X(t) = E\left(e^{tX}\right) = \int_0^\infty \left(e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\right) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(x^{\alpha-1} e^{-(\lambda-t)x}\right) dx$$

$$M_X(t) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^\infty \left(\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}\right) dx$$

The integrand is pdf of *Gamma* ($\alpha, \lambda - t$) and the value of integral is 1.

Therefore, $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$, provided $t < \lambda$.

Dividing the numerator and denominator by λ gives:

$$M_X(t) = \left(\frac{1}{1-\frac{t}{\lambda}}\right)^\alpha = \left(1-\frac{t}{\lambda}\right)^{-\alpha} \qquad t < \lambda$$

The cumulant generating function is:

$$C_X(t) = \log M_X(t) = -\alpha \log\left(1-\frac{t}{\lambda}\right)$$

ii) The coefficient of skewness $= \frac{Skew(X)}{Var(X)^{1.5}}$

We know that $Var(X) = C_X''(0)$ and $Skew(X) = C_X'''(0)$

$$C_X'(t) = \frac{\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-1}$$

$$C_X''(t) = \frac{\alpha}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-2} \Rightarrow \quad Var(X) = C_X''(0) = \frac{\alpha}{\lambda^2}$$

$$C_X'''(t) = \frac{2\alpha}{\lambda^3}\left(1-\frac{t}{\lambda}\right)^{-3} \Rightarrow \quad Skew(X) = C_X'''(0) = \frac{2\alpha}{\lambda^3}$$

Hence, the coefficient of skewness $= \frac{2\alpha/\lambda^3}{\left(\alpha/\lambda^2\right)^{1.5}} = \frac{2\alpha}{\alpha^{1.5}} = \frac{2}{\sqrt{\alpha}}$

5.

The moment generating function of $X_1 = M_{X_1}(t_1) = E(e^{t_1 x_1})$

$$= M_X(t_1, 0) \ = \frac{1}{3}\left(1 + e^{(t_1 + 2\times 0)} + e^{(2t_1 + 0)}\right)$$

$$= \frac{1}{3}\left(1 + e^{t_1} + e^{2t_1}\right)$$

The expected value of $X_1$is obtained bytaking first derivative of its MGF and evaluating

at t_1=0.

Thus, $$E(X_1) = M_{X_1}(0) = \frac{1}{3}\left(1 + e^0 + e^0\right) = 1$$

Similarly using the mgf of X_2 , $E(X_2)$ is shown to be 1

$E(X_1X_2)$ is computed by taking the second cross-partial derivative of joint moment generating function evaluated at $(t_1, t_2) = (0,0)$:

$$\frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2}\left(\frac{1}{3}[1 + \exp(t_1 + 2t_2) + \exp(2t_1 + t_2)]\right)\right)$$

$$= \frac{\partial}{\partial t_1}\left(\frac{1}{3}[2\exp(t_1 + 2t_2) + \exp(2t_1 + t_2)]\right)$$

$$= \frac{1}{3}[2\exp(t_1 + 2t_2) + 2\exp(2t_1 + t_2)]$$

Thus, $E(X_1X_2) = \frac{4}{3}$

$$Cov(X_1X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{4}{3} - 1 \times 1 = \frac{1}{3} = 0.33$$

6.

i) The marginal *pdf* of Y:

i. $f_1(y) = \int_0^y 2\, dx = \begin{cases} 2y & 0 < y < 1 \\ 0, & otherwise. \end{cases}$

ii) The conditional *pdf* of X given Y: $f(x/y)$ is

i. $f(x/y) = \frac{f(x,y)}{f_1(y)} = \begin{cases} \frac{2}{2y}\, ;\ 0 < x < y;\ 0 < y < 1 \\ 0\ otherwise. \end{cases}$

iii) The conditional mean: $E(X/Y = 2) = \int_0^y x\, \frac{1}{y} dx = \frac{y}{2},\ 0 < y < 1$

1. $= \frac{1}{4} = 0.25$ when $y = \frac{1}{2}$

iv) The conditional variance: $V(X/\ Y = y) = E(X^2/Y = y) - (E(X/Y = 2))^2$

a. $E(X^2/Y = y) = \int_0^y x^2 \frac{1}{y}\, dx = \frac{y^2}{3}\ ; 0 < y < 1$

1. $= \frac{1}{12} = 0.083$ when $y = \frac{1}{2}$.

Hence, $V(X/\ Y = \frac{1}{2}) = \frac{1}{12} - (\frac{1}{4})^2 = \frac{1}{48} = 0.0208$.

7.

The mean and variance of the Pareto distribution are given by:

$$E\,[X] = \frac{\lambda}{\alpha - 1} = \frac{4}{3 - 1} = 2$$

$$Var\,[X] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3\,(4)^2}{(3-1)^2(3-2)} = 12$$

From page 16 of the *Tables* $Var\,(Y) = Var\,[E\,(Y \mid X)] + E\,[Var\,(Y \mid X)]$:

$$Var\,(Y) = Var\,[2X + 5\,] + E\,[X^2 + 3]$$

$Var\ (Y) = 4\, Var\ [X] + E\ [X^2] + 3$

We know that $E\ [X^2] = Var[X] + (E\ [X])^2 = 12 + 2^2 = 16$
$Var\ (Y) = 4\ (12) + 16 + 3 = 67$

Hence, the standard deviations is $\sqrt{67} = 8.1854$

8.

(i) The Joint density of *(X,Y)*:
We know that $f(y|x) = \frac{f(x,\ y)}{f(x)}$.
Hence, $f(x,y) = f(y|x)f(x)$

$= \frac{1}{x}\ (8x)$

Thus, $f(x,y) = \begin{cases} 8 & if\ 0 < x < \frac{1}{2}\ and\ 0 < y < x \\ 0 & Otherwise. \end{cases}$

(ii) The marginal density of *Y*:

$f(y) = \int f(x,y)dx = \int_y^{0.5} 8\, dx = 4(1-2y)\ \ for\ 0<y<\frac{1}{2}$.

(iii) The Mean and Variance of *X* and *Y*:

$E(Y) = \int_0^{0.5} (y)\ 4\ (1-2y)\ dy = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$.

$V(Y) = E(Y^2) - (E(Y))^2$.

$E(Y^2) = \int_0^{0.5} y^2\ 4(1-2y)\ dy = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}$.

$V(Y) = \frac{1}{24} - \left(\frac{1}{6}\right)^2 = \frac{1}{72}$.

9.

(i)

Given that the mean of the binomial distribution is (np) =12 and n=20.

Hence p=0.6 .That is the distribution is binomial (20, 0.6)

The PGF of binomial distribution is given by

$G_X(t) = E(t^X) = \sum_{x=0}^{20} t^x P(X=x)$

$= \sum_{x=0}^{20} t^x \binom{20}{x} (0.6)^x (1-0.6)^{(20-x)}$

$= \sum_{x=0}^{20} \binom{20}{x} (0.6\,t)^x (1-0.6)^{(20-x)}$

$= (0.6\,t + (1-0.6))^{20}$

$= (0.4 + 0.6\,t)^{20}$

For MGF, replace t by e^t in the above expression

$M_X(t) = (0.4 + 0.6\,e^t)^{20}$

10.

$V(Y) = E[V(Y|X)] + V[E(Y|X)]$

$= E(X+1) + V(2X+3)$

$= E(X) + 1 + 4V(X)$

$= 5 + 1 + 4(5)$

$= 26$

11. i) Option C

ii) $M_x(t) = E(e^{tx}) = \int_a^\infty e^{tx} \lambda e^{-\lambda(x-a)} dx$

$= \int_a^\infty e^{\lambda a} \lambda e^{-x(\lambda - t)} dx$

$= e^{\lambda a} \lambda e^{-x(\lambda-t)} / -(\lambda - t)|_a^\infty$

$= \lambda/(\lambda - t)\, e^{at}$, Provided $t<\lambda$

$= (1-t/\lambda)^{-1} e^{at}$

iii) $M_x(t) = (1-t/\lambda)^{-1} e^{at}$

$M'_x(t) = (1-t/\lambda)^{-1} e^{at} a + (1-t/\lambda)^{-2} 1/\lambda\, e^{at}$

$E(x) = M'_x(0) = a + 1/\lambda$

iv) Option B

12.

i) The marginal density is

$f_Y(y) = 3\int_0^{\infty} e^{-x}e^{-3y}dx = 3e^{-3y}\int_0^{\infty} e^{-x}dx = 3e^{-3y}$

ii) The conditional probability P(Y ≤ y | Y> 4) is $F_Y(y)$

$$P(Y \le y \mid Y>4) = \frac{P(Y \le y, Y>4)}{P(Y>4)} = \frac{P(4<Y \le y)}{P(Y>4)} = \frac{P(4<Y \le y)}{P(Y>4)} = \frac{FY(y)-FY(4)}{P(Y>4)}, y>4$$

Therefore

$$f(y \mid Y>4) = \frac{fY(y)}{P(Y>4)} = \frac{3e^{-3y}}{e^{-12}} = 3e^{12-3y}, \ y>4$$

iii) The correct option is (C)

The conditional expectation is given as

$$E[Y \mid Y>4] = \int_4^{\infty} yf(y \mid Y>4)dy = \int_4^{\infty} 3ye^{12-3y}dy$$

By taking t = y-4,

$$E[Y \mid Y>4] = \int_0^{\infty} 3(t+4)e^{-3t}dt = \int_0^{\infty} 3te^{-3t}dt + \int_0^{\infty} 12e^{-3t}dt$$