INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability and Statistics – I

Chapter: Unit 4

Category: Practice Questions

1. **CS1 April 2021 Question 2**
Consider two random variables, X and Y. The conditional expectation and conditional variance of Y given X are denoted by the two random variables U and V, respectively; that is,
$$U = E[Y|X] \text{ and } V = Var[Y|X].$$

Assume that Y is Normally distributed with expectation 5 and variance 4. Also assume that the expectation of V is 2.
i. Calculate the expected value of U.
ii. Calculate the variance of U.

2. **CS1 April 2021 Question 5**
The joint probability density function of random variables X and Y is:

$$f(x,y) = \begin{cases} ke^{-(x+2y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

[Hint: You may find it helpful to define the functions $g_X(x) = e^{-x}$ and $g_Y(y) = e^{-2y}$, using this notation in your answers]

i. Demonstrate that X and Y are independent.
ii. Verify that k = 2
iii. Demonstrate that $f_Y(y)$, the marginal density function of Y, is:
$$2e^{-2y} \text{ for } y > 0.$$

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3. **CS1 September 2019 Question 4**
X and Y are discrete random variables with joint distribution as follows:

	$X = 0$	$X = 1$	$X = 3$
$Y = -1$	0.08	0.03	0.00
$Y = 0$	0.03	0.12	0.20
$Y = 3$	0.11	0.11	0.06
$Y = 4.5$	0.04	0.20	0.02

i. Calculate:
(a) E(Y | X = 1)
(b) Var(X | Y = 3).

ii. Calculate the probability functions of the marginal distributions for X and Y.

iii. Determine whether X and Y are independent.

4. **CS1 April 2019 Question 6**
Let X and Y be two continuous random variables.

i. State the definition of independence of the random variables X and Y in terms of their joint probability density function.

The joint probability density function of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii. (a) Determine the marginal density functions of X and Y.
(b) State whether or not X and Y are independent based on your answer in part (ii)(a).

iii. Derive the conditional expectation E[X | Y = y].

5. CT3 April 2018 Question 9

The random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 24x^3y & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

i. (a) Show that the marginal pdf of X is

$$f_X(x) = 12x^3(1-x^2),\ 0 < x < 1.$$

(b) Show that the marginal pdf of Y is $f_Y(y) = 6y^5$, 0 < y < 1.

ii. Determine the covariance cov (X, Y).

iii. Determine the conditional pdf $f_{(Y)}$(x| y) together with the range of X for which it is defined.

iv. Determine the conditional probability $P\left(X > 1/3 \mid Y = \frac{1}{2}\right)$

v. Determine the conditional expectation $E\left(X \mid Y = \frac{1}{4}\right)$

vi. Verify that E[E[X |Y]] = E[X] by evaluating each side of the equation.

6. CT3 April 2017 Question 3

Consider two random variables X and Y and assume that X and Y both follow a standard normal distribution but are not independent. Define the random variables:

$$Z^- = X - Y \text{ and } Z^+ = X + Y.$$

i. Determine the covariance between Z− and Z+

ii. Determine whether Z− and Z+ are uncorrelated based on your answer in part (i).

7. CT3 September 2016 Question 6

Let X and Y be random variables with joint probability distribution:

$$f_{XY}(x,y) = \begin{cases} kx^2y^2, & 0 < x < y < 1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

i. Show that k = 18.

ii. Determine $f_Y(y)$, the marginal density function of Y.
iii. Determine P(X > 0.5|Y = 0.75).

8. CT3 September 2015 Question 7

X and Y are discrete random variables with joint distribution given below.

	$Y=-1$	$Y=0$	$Y=1$
$X=1$	0	1/4	0
$X=0$	1/4	1/4	1/4

i. Determine the conditional expectation E[Y|X = 1].
ii. Determine the conditional expectation E[X|Y = y] for each value of y.
iii. Determine the expected value of X based on your conditional expectation results from part (ii).

9. CT3 April 2015 Question 8

The random variables X and Y have a joint probability distribution with density function:

$$f_{xy}(x,y)=\begin{cases}3x, & 0<y<x<1\\ 0, & \text{otherwise}\end{cases}$$

i. Determine the marginal densities of X and Y.
ii. State, with reasons, whether X and Y are independent.
iii. Determine E[X] and E[Y].

10. CT3 September 2014 Question 5

Consider two random variables X and Y with E[X] = 2, V[X] = 4, E[Y] = −3, V[Y] = 1, and Cov[X, Y] = 1.6.

Calculate:

a. the expected value of 5X + 20Y.
b. the correlation coefficient between X and Y.
c. the expected value of the product XY.
d. the variance of X − Y.

11. CT3 April 2014 Question 7

Let X and Y be two continuous random variables.

i. Prove that E[E[Y|X]] = E[Y].

Suppose the number of claims, N, on a policy follows a Poisson distribution with mean μ, and the amount of the ith claim, X_i , follows a Gamma distribution with parameters α and λ. Let S denote the total value of claims on a policy in a given year.

ii. Derive the mean of S using the result in part (i).

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Suppose μ = 0.15, α = 100, and λ = 0.1.

iii. Calculate the variance of S.

12. CT3 September 2012 Question 4

Consider a random variable U that has a uniform distribution on (0,1) and a random variable X that has a standard normal distribution. Assume that U and X are independent.

Determine an expression for the probability density function of the random variable Z = U + X in terms of the cumulative distribution function of X.

13. CT3 April 2012 Question 10

In a portfolio of car insurance policies, the number of accident-related claims, N, made by a policyholder in a year has the following distribution:

No. of claims, n	0	1	2
Probability	0.4	0.4	0.2

The number of cars, X, involved in each accident that results in a claim is distributed as follows:

No. of cars, x	1	2
Probability	0.7	0.3

It can be assumed that the occurrence of a claim and the number of cars involved in the accident are independent.

Furthermore, claims made by a policyholder in any year are also independent of each other.

Let S be the total number of cars involved in accidents related to such claims by a policyholder in a year.

i. (a)Determine the probability function of S.
(b)Hence find E(S).

The expectation E(S) can also be calculated using the formula

$$E(S) = \sum_{n=0}^{2} E(S|N=n)\Pr(N=n).$$

ii. (a) Find E(S|N = n) for n = 0,1 , 2 .
(b) Hence calculate E(S).

14. CS1A April 2022 Q1

UNIT 4

PRACTICE QUESTIONS

The number of emails, X, to be replied to in a day by an employee of the customer service centre of an insurance company is modeled as a Poisson random variable with mean 25.

The time (in minutes), Y, that the employee takes to reply to x emails is modeled as a random variable with conditional mean and variance given by:

$E(Y|X = x) = 3x + 11$ and $Var(Y|X = x) = x + 9$.

Calculate the unconditional variance of the time, Y, that the employee takes to reply to emails in a day. [3]

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