INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability and Statistics – I

Chapter: Unit 4

Category: Practice Questions

1. CT3 September 2012 Question 4

Consider a random variable U that has a uniform distribution on (0,1) and a random variable X that has a standard normal distribution. Assume that U and X are independent.
Determine an expression for the probability density function of the random variable Z = U + X in terms of the cumulative distribution function of X.

Ans: $F_X(z)-F_X(z-1)$
where we have used the substitution $u = z - x$, and where F_X is the distribution function of X.

2. CT3 April 2014 Question 7

Let X and Y be two continuous random variables.

i. Prove that E[E[Y|X]] = E[Y].

Suppose the number of claims, N, on a policy follows a Poisson distribution with mean μ, and the amount of the ith claim, X_i, follows a Gamma distribution with parameters α and λ. Let S denote the total value of claims on a policy in a given year.

ii. Derive the mean of S using the result in part (i).

Suppose μ = 0.15, α = 100, and λ = 0.1.

iii. Calculate the variance of S.

Ans:
(i) –
(ii) E(S)=$\mu\alpha/\lambda$
(iii) V(S)= 151,500

3. CT3 September 2014 Question 5

Consider two random variables X and Y with E[X] = 2, V[X] = 4, E[Y] = −3, V[Y] = 1, and Cov[X, Y] = 1.6.

Calculate:

(i) the expected value of 5X + 20Y.

(ii) the correlation coefficient between X and Y.

(iii) the expected value of the product XY.

(iv) the variance of X − Y.

Ans:
(i) E(5X + 20Y)=-50
(ii) Corr(X,Y)=0.8
(iii) E(XY)=-4.4
(iv) V(X-Y)=1.8

4. CT3 April 2015 Question 8
The random variables X and Y have a joint probability distribution with density function:

$$f_{xy}(x,y)=\begin{cases}3x, & 0<y<x<1\\ 0, & \text{otherwise}\end{cases}$$

i. Determine the marginal densities of X and Y.

ii. State, with reasons, whether X and Y are independent.

iii. Determine E[X] and E[Y].

Ans:
(i)

$$f_X(x)=\int_0^x 3x\,dy=[3xy]_{y=0}^{y=x}=3x^2 \text{ for } 0<x<1$$

$$f_Y(y)=\int_y^1 3x\,dx=\left[\frac{3}{2}x^2\right]_{x=y}^{x=1}=\frac{3}{2}\left(1-y^2\right) \text{ for } 0<y<1$$

(ii) Not independent because $f_X(x)f_Y(y)\neq f_{XY}(x,y)$
(iii) E(X)=0.75, E(Y)= 3/8

5. CT3 September 2015 Question 7

X and Y are discrete random variables with joint distribution given below.

	$Y=-1$	$Y=0$	$Y=1$
$X=1$	0	1/4	0
$X=0$	1/4	1/4	1/4

i. Determine the conditional expectation E[Y|X = 1].

ii. Determine the conditional expectation E[X|Y = y] for each value of y.

iii. Determine the expected value of X based on your conditional expectation results from part (ii).

Ans:

(i) E[Y|X = 1] = 0

(ii) E[X|Y = -1] = 0, E(X|Y=0)=1/2, E(X|Y=1)=0

(iii) E(X)=1/4

6. CT3 September 2016 Question 6

Let X and Y be random variables with joint probability distribution:

$$f_{XY}(x,y)=\begin{cases} kx^2y^2, & 0<x<y<1 \\ 0, & \text{otherwise} \end{cases}$$

where k is a constant.

i. Show that k = 18.

ii. Determine $f_Y(y)$, the marginal density function of Y.

iii. Determine P(X > 0.5|Y = 0.75).

Ans:

(i) –

(ii) $f_Y(y)=6y^5$

(iii) P(X > 0.5|Y = 0.75)= 0.7037

7. CT3 April 2017 Question 3

Consider two random variables X and Y and assume that X and Y both follow a standard normal distribution but are not independent. Define the random variables:

$$Z^{-}=X-Y \text{ and } Z^{+}=X+Y$$

i. Determine the covariance between Z^{-} and Z^{+}

ii. Determine whether Z− and Z+ are uncorrelated based on your answer in part (i).

Ans:

(i) $Cov(Z^{-}, Z^{+}) = 0$

(ii) Since $corr(Z^{-}, Z^{+}) = cov(Z^{-}, Z^{+}) / \{sd(Z^{-})\, sd(Z^{+})\}$ it follows that Z^{-} and Z^{+} are uncorrelated.

8. CT3 April 2018 Question 9

The random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 24x^3y & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

i. (a) Show that the marginal pdf of X is

$$f_X(x) = 12x^3(1-x^2),\ 0 < x < 1.$$

(b) Show that the marginal pdf of Y is $f_Y(y) = 6y^5$, $0 < y < 1$.

ii. Determine the covariance cov (X, Y).

iii. Determine the conditional pdf $f_{(Y)}(x|\, y)$ together with the range of X for which it is defined.

iv. Determine the conditional probability $P\left(X > 1/3 \mid Y = \frac{1}{2}\right)$

v. Determine the conditional expectation $E\left(X \mid Y = \frac{1}{4}\right)$

vi. Verify that E[E[X |Y]] = E[X] by evaluating each side of the equation.

Ans:

(i) –

(ii) Cov(X,Y)=3/245

(iii) $f_{X|Y}(x|y)=4x^3y^{-4}$

(iv) $P\left(X > 1/3 \mid Y = \frac{1}{2}\right)$=65/81

(v) $E\left(X \mid Y = \frac{1}{4}\right)$= (4/5)y

(vi) -

9. CS1 April 2019 Question 6

Let X and Y be two continuous random variables.

i. State the definition of independence of the random variables X and Y in terms of their joint probability density function.

The joint probability density function of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 8xy & \text{for } 0 < x < y < 1, \\ 0 & \text{otherwise.} \end{cases}$$

ii. (a) Determine the marginal density functions of X and Y.
(b) State whether or not X and Y are independent based on your answer in part (ii)(a).

iii. Derive the conditional expectation E[X | Y = y].

Ans:

(i) The random variables *X* and *Y* are independent if, and only if, the joint pdf is the product of the two marginal pdfs for all (*x*,*y*) in the range of the variables, i.e.
$f_{XY}(x,y)=f_X(x)*f_Y(y)$ for all (*x*, *y*) in the range.

(ii) (a) $f_X(x)= 4x(1-x^2)$, $0<x<1$
$f_Y(y) = 4y^3$, $0 < y < 1$
(b) Here, $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ so *X* and *Y* are not independent.

(iii) $E(X|Y=y)$=(2/3)y, 0<y<1

10. CS1 September 2019 Question 4

X and Y are discrete random variables with joint distribution as follows:

	$X = 0$	$X = 1$	$X = 3$
$Y = -1$	0.08	0.03	0.00
$Y = 0$	0.03	0.12	0.20
$Y = 3$	0.11	0.11	0.06
$Y = 4.5$	0.04	0.20	0.02

i. Calculate:
(a) E(Y | X = 1)
(b) Var(X | Y = 3).

ii. Calculate the probability functions of the marginal distributions for X and Y.

iii. Determine whether X and Y are independent.

Ans:

(i) (a) E(Y | X = 1) =2.6087
(b) Var(X | Y = 3) =1.2487

(ii) $P(X = 0)$= 0.26, $P(X = 1)$= 0.46, $P(X=3)$ = 0.28
$P(Y = -1)$= 0.11, $P(Y = 0)$= 0.35, $P(Y = 3)$= 0.28, $P(Y = 4.5)$ = 0.26

(iii) $P(X = 0, Y = -1)$ = 0.08≠ $P(X = 0) \times P(Y = -1)$
Correct conclusion that X and Y are NOT independent

11. CS1 April 2021 Question 2

Consider two random variables, X and Y. The conditional expectation and conditional variance of Y given X are denoted by the two random variables U and V, respectively; that is,

$$U = E[Y|X] \text{ and } V = Var[Y|X].$$

Assume that Y is Normally distributed with expectation 5 and variance 4. Also assume that the expectation of V is 2.

i. Calculate the expected value of U.

ii. Calculate the variance of U.

Ans:

(i) E(U)=E(Y)=5

(ii) V(U)=2

12. CS1 April 2021 Question 5

The joint probability density function of random variables X and Y is:

$$f(x,y) = \begin{cases} ke^{-(x+2y)}, & x > 0, y > 0 \\ 0, & \text{otherwise} \end{cases}$$

[Hint: You may find it helpful to define the functions $g_X(x) = e^{-x}$ and $g_Y(y) = e^{-2y}$, using this notation in your answers]

i. Demonstrate that X and Y are independent.

ii. Verify that k = 2

iii. Demonstrate that $f_Y(y)$, the marginal density function of Y, is:

$$2e^{2y} \text{ for } y>0$$

iv. Demonstrate that the conditional density function $f(y|Y > 3)$ is:

$$f(y|Y > 3) = 2e^{6-2y} \text{ for } y > 3.$$

[**Hint:** Consider $P(Y \leq y|Y > 3)$.]

v. Identify which **one** of the following expressions is equal to the conditional expectation $E[Y|Y > 3]$:

A $\int_0^\infty te^{-2t}dt + \int_0^\infty 3e^{-2t}dy$

B $\int_0^\infty te^{-2t}dt + \int_0^\infty 6e^{-2t}dy$

C $\int_0^\infty 2te^{-2t}dt + \int_0^\infty 3e^{-2t}dy$

D $\int_0^\infty 2te^{-2t}dt + \int_0^\infty 6e^{-2t}dy$

vi. Determine the value of the conditional expectation $E[Y|Y > 3]$.

vii. Identify which **one** of the following options is the conditional expectation $E[Y2|Y > 3]$:
A 12.5
B 13.5
C 14.5
D 15.5

viii. Determine the conditional variance $\text{Var}[Y|Y > 3]$.

Ans:
(i) $f x,y) = ke^{-x}e^{-2y}$, $x>0$, $y>0$
The density function is expressed as a product of a function of x and y. Therefore, the joint probability function is a product of the two marginal probability functions for all (x,y) in the range of the variables hence $X$and Y are independent
(ii) –
(iii) –
(iv) –
(v) D
(vi) $E[Y|Y>3]=3.5$
(vii) D
(viii) $\text{Var}[Y|Y > 3]=0.25$

14. CS1A April 2022 Q1

The number of emails, X, to be replied to in a day by an employee of the customer service centre of an insurance company is modeled as a Poisson random variable with mean 25.

The time (in minutes), Y, that the employee takes to reply to x emails is modeled as a random variable with conditional mean and variance given by:

$E(Y|X = x) = 3x + 11$ and $\text{Var}(Y|X = x) = x + 9$.

Calculate the unconditional variance of the time, Y, that the employee takes to reply to emails in a day.

Ans: $Var\ (Y)$ =259

15. CS1A April 2024 Q1

The following table shows possible realisations, x, of a random variable, X, the probabilities for these realisations and the conditional expectations and variances of a random variable, Y, conditionally on $X = x$.

Realisation (x)	10	20	40
P(X=x)	0.4	0.4	0.2
E[Y|X=x]	2	8	10
V[Y|X=x]	1	2	3

(i) Identify which one of the following options gives the correct values of the unconditional expectations of X and Y:

A. E(X)=20 and E(Y)=6

B. E(X)=20 and E(Y)=4

C. E(X)=24 and E(Y)=6

D. E(X)=24 and E(Y)=4.

(ii) Calculate the unconditional variance of Y.

Ans:

(i) A

(ii) (ii) V(Y) = 13

16. CS1 April 2024 Q3

X and *Y* are discrete random variables with joint distribution given in the table below.

	Y = -3	**Y = -2**	**Y = -1**	**Y = 0**
X = 1	2/5	0	1/10	0
X = 2	1/5	1/10	0	1/5

(i) Identify which **one** of the following options gives the correct value of the conditional expectation E[Y|X = 2]

A −2.2

B −1.8

C −1.6

D −0.3. [2]

(ii) Determine the conditional expectation E[X | Y = y] for each value of y. [4]

(iii) Calculate the expected value of X using your result from part (ii). [2]

Ans:

(i) C

(ii)

	Y = -3	**Y = -2**	**Y = -1**	**Y = 0**
E[X | Y =y]	4/3	2	1	2

(iii) E(X) = 3/2]