INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability & Statistics

Chapter: Unit 1 & 2

Category: Assignment Solutions

1.

i) kth moment = $E[X^k]$

ii) kth moment about α. $= E[(X-\alpha)^k]$

iii) kth central moment $= E[(X-\mu)^k]$ where μ is the mean

iv) Coefficient of skewness $= \frac{E[(X-\mu)^3]}{\sigma^3}$ where σ is the std. deviation

2.

i) $\text{P[Withdrawal]} = \text{P[Withdrawal | Agency]. P[Agency]} +$
$\text{P[Withdrawal | Bank]. P[Bank]} + \text{P[Surrender | Online]. P[Online]}$
$= 0.05\left(\frac{2000}{10000}\right) + 0.08\left(\frac{3500}{10000}\right) + 0.14\left(\frac{4500}{10000}\right) = 0.101$

ii) $\text{P[Withdrawal} \cap \text{Agency]} = \text{P[Withdrawal | Agency]. P[Agency]} = 0.05 * 0.20 = 0.01$
$\text{P[Withdrawal} \cap \text{Bank]} = \text{P[Withdrawal | Bank]. P[Bank]} = 0.08 * 0.35 = 0.028$
$\text{P[Withdrawal} \cap \text{Not online]} = 0.01 + 0.028 = 0.038$
P[Withdrawal | Not Online] = P[Withdrawal ∩ Not Online] / P[Not Online]
$= \frac{0.038}{1-0.45} = 0.0691$

iii) $P[Bank \mid \text{Withdrawal}] = \frac{P[\text{Withdrawal} \cap Bank]}{P[\text{Withdrawal}]} = \frac{0.280}{0.101} = 0.2772$

3.

i) Ordering the marks given, stem and leaf diagram is:

1	7, 9
2	1, 1, 2, 2, 6, 6, 7, 7, 7, 7
3	0, 0, 1, 2, 3, 3, 3, 4, 6
4	6
5	3
6	0

The stems are 10s and leaves are units.

ii) Median: $\left(\frac{1}{2}n + \frac{1}{2}\right)^{th}$ value $=12.5^{\text{th}}$ value =(27+30)/2 =28.5.

Mode: 27. (27 appears the maximum number of times-four times)

iii) Interquartile Range (IQR) = $Q_3 - Q_1$

Now Q_1 = $\frac{n+2}{4}$ th value counting from below = 6.5^{th} value

= (22+26)/2 = 24

Q_1 = $\frac{n+2}{4}$ th value counting from above =33

Hence, $IQR = Q_3 - Q_1$ =33-24 =9

[*Alternatively,*

Q_1 = $\frac{n+1}{4}$ *th value counting from below = 6.25th value =23 and*

Q_3= $\frac{n+1}{4}$ *th value counting from above= 33*

Hence IQR = 33-23=10]

4.

The binomial distribution (n, p) has probability function.

$$P(X=x) = \frac{n!}{(n-x)!\,x!} p^x (1-p)^{(n-x)} \;;\quad x=0,1,2,\ldots;\ 0<p<1.$$

and $P(X=x-1) = \frac{n!}{(n-x+1)!(x-1)!} p^{(x-1)} (1-p)^{(n-x+1)}$; $x=1,2,3,\ldots$;$0<p<1$

Now, $\frac{P(X=x)}{P(X=x-1)} = \frac{n-x+1}{x}\frac{p}{1-p}$

Therefore, $P(X=x) = \frac{n-x+1}{x}\frac{p}{1-p} P(X=x-1)$; $x=1,2,3,\ldots$

5.

i) Let Success: = Getting one passenger to go to *B* from *A*.

p = Probability [Getting one passenger to go to *B* from *A*.] = 0.3;

$q = 1 - p = 0.7$

k = the number of successes (number of passengers going to Town *B*) = 4

$X+k$ = the number of trials=15

X = the number of failures = 11

X follows *NB* ($k=4$, $p = 0.3$)

$$[X \sim NB(k,p) \rightarrow P(X=x) = \binom{x+k-1}{x} p^k q^x; = x = 0,1,\ldots \qquad]$$

Hence, $P(X=11) = \binom{11+4-1}{11} 0.3^4 0.7^{11}$

$$= \frac{14!}{11!3!} 0.3^4 0.7^{11} = 0.0583.$$

ii) The average number of persons to be asked in order to get 4 passengers

= E (the number of trials)

$= E(X+k) = E(X) + k = \frac{kq}{p} + k = \frac{k}{p}$

$= \frac{4\times 0.7}{0.3} + 4 = \frac{4}{0.3}$

$= 13.33 \sim 14$ persons.

6.

i) Mode: For fixed $\vartheta > 0$, the density function $f(x)$ is an increasing function of x.

Thus, $f(x)$ has maximum at the right end point of the interval $[0, \vartheta]$.

Hence the mode of this distribution is ϑ.

Median:

$$\tfrac{1}{2} = \int_0^q f(x)\,dx$$

$$= \int_0^q \frac{3x^2}{\theta^3}\,dx = [x^3/\theta^3] \text{ from 0 to } q$$

Thus, $\tfrac{1}{2} = q^3/\theta^3$ implies $q = \frac{\theta}{2^{\frac{1}{3}}}$

ii) Let A be the ratio of the mode of this distribution to the median

A = mode/median = $\theta \times \frac{2^{\frac{1}{3}}}{\theta} = 2^{\frac{1}{3}} = 1.2599$

$$P(X < A) = \int_0^A f(x)\,dx$$

$$= \int_0^A 3x^2/\theta^3\,dx$$

$$= [x^3/\theta^3] \text{ from 0 to } A$$

$$= A^3/\theta^3$$

$$= \begin{cases} \frac{2}{\theta^3} & if \quad \theta > 2^{1/3} = 1{\cdot}2599 \\ 1 & otherwise \end{cases}$$

7.

i) Median = $\left(\frac{1}{2}n + \frac{1}{2}\right)^{th}$ $value$

$Q_1 = \left(\frac{1}{4}n + \frac{1}{2}\right)^{th}$ value and $Q_3 = \left(\frac{3}{4}n + \frac{1}{2}\right)^{th}$ value.

For city A:

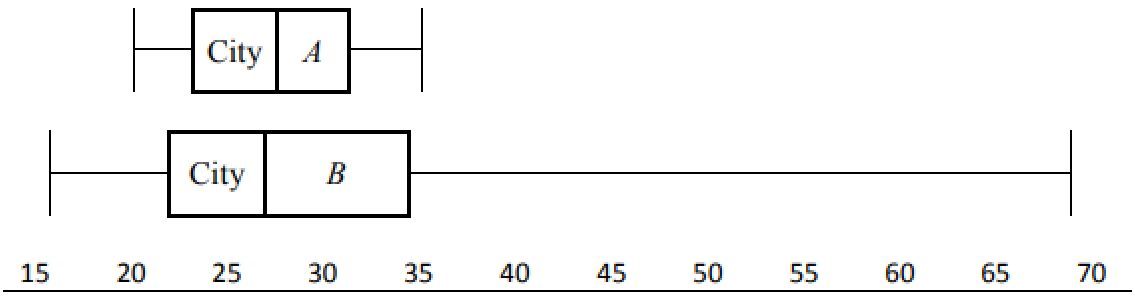
Median = 27.50

$Q_1 = 23.50$; $Q_3 = 31.50$

For city B:

Median = 27.00

$Q_1 = 22.50$; $Q_3 = 34.50$



ii) Looking at the box plots, we see that the median of both distributions are close to 27 ° Celsius. This suggests that the monthly maximum temperatures for City A and B may have the averages close to each other.

However, the overall spread of the figures for city B appears to be greater than the corresponding spread for city A which can be confirmed by measuring by IQR. This suggests that the variability in the monthly maximum temperatures for City B is greater than the corresponding variability for city A (although conclusions drawn from such small sample sizes should be treated with caution).

The value of 68 for city B could be an outlier.

City A distribution seems symmetric and City B distribution is clearly positively skewed.

For comparison:

- for city A: the modes (26.00) ≈ median (27.50) = mean (27.50)
- for city B: the mode (25.00) < median (27.00) < mean (30.75)

8.

The χ_9^2 distribution has mean 9 and variance 18.

Setting the expressions for the mean and variance of the lognormal distribution to the above values and solving the two equations:

$$e^{\mu + \frac{1}{2}\sigma^2} = 9\,; \qquad\qquad e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right) = 18$$

Squaring the first equation and substituting into the second, we get:

$$81\left(e^{\sigma^2} - 1\right) = 18$$

Solving this, we get $\sigma^2 = \log\frac{99}{81} = 0.2007$

Now substituting σ^2in any one equation, we get $\mu = 2.0969$.

If $X \sim$ log Normal then $\log X \sim$ Normal, so: $P\ (X > 9) = P\ (\log X > \log 9)$

$$= \mathrm{P}\ \left(Z > \frac{\log 9 - 2.0969}{\sqrt{0.2007}}\right) = \mathrm{P}\ (Z > \ 0.224)$$

Using interpolation from tabulated values for 0.22 and 0.23 in page 160

$$= 1 - P\ (Z < 0.224) = 1 - 0.5886 = 0.4114$$

[5 Marks]

9.

Let X be the time for the first exotic cake and Y be the time for the second. Then:

$X \sim N(120, 225)$ and $Y \sim N(120, 225)$ [in terms of minutes]

We require: $P(|X - Y| < 25)$

The distribution of X - Y, is:

$$(X - Y) \sim N(120 - 120, 225 + 225) \sim N(0, 450)$$

$$P(|X - Y| < 25) = P(-25 < X - Y < 25)$$

$$= P\left(\frac{-25}{\sqrt{450}} < Z < \frac{25}{\sqrt{450}}\right)$$

$$= P(Z < 1.1785) - P(Z < -1.1785)$$
$$= P(Z < 1.1785) - (1 - P(Z < 1.1785))$$

$$= 2P(Z < 1.1785) - 1$$
$$= (2)(0.8807) - 1 = 0.7614$$

10.

(i) Let S be the salary (constant) of each of the 99 employees.

Hence, average salary of 99 employees = (S+S+...+S)/99 = S
The variance of the salary of 99 employees was 0 (as they are getting the constant salary)

[1]

(ii) S+1000 is the salary of the 100th employee.

So, the average salary of 100 employees = [(S+S+...+S) +(S+1000)] /100
= (100S+1000)/100 = S+10

With the addition of the 100th employee, the average salary of 100 employees has increased by Rs 10.

Now, the variance of the salary of 100 employees

$$\frac{1}{100}[\{(-10)^2+(-10)^2+...+(-10)^2\}+990^2]$$

$$= \frac{1}{100}[9900+980100] = \frac{1}{100}(990000) = 9900$$

The standard deviation = $\sqrt{9900}$ = Rs 99.50, which is positive

11.

(i) Let *T*: Accident with tyre burst

C: Accident due to collision with the road divider and

D: Death casualty in a car accident

Given that: $P(T) = 0.6$, $P(C) = 0.4$, $P(D|T) = 0.3$ and $P(D|C) = 0.5$.

Given accidental death casualty, the probability of tyre burst is:

$$P(T|D) = \frac{P(T)P(D|T)}{P(T)\,P(D|T) + P(C)\,P(D|C)}$$

$$= \frac{0.6\,(0.3)}{(0.6\,(0.3) + 0.4\,(0.5))} = \frac{18}{38} = 0.4737.$$

(ii) To find the most probable cause of death casualty due to accidents, we need to compare given accidental death casualty, the probability of tyre burst and probability of collision with the road divisor; i.e $P(T|D)$ and $P(C|D)$

Now, $P(C|D) = \frac{P(C)P(D|C)}{P(T)\,P(D|T) + P(C)\,P(D|C)}$

$$= \frac{0.4\,(0.5)}{(0.6\,(0.3) + 0.4\,(0.5))} = \frac{20}{38} = 0.5263.$$

[Alternately, $P(C/D) = 1 - P(T/D) = \frac{20}{38}$]

Since, $P(T|D) < P(C|D)$, collision with the road divider is the most probable cause of accidental death casualty.

12.

(i) P(obtaining correct password in the third try)
= P(obtaining incorrect password in the first two attempts and obtaining correct password in the third attempt)
= (1- 1/100) (1- 1/99) (1/98)
=1/100. **[2]**

(ii) File can be accessed if the password is correct in the first attempt or second attempt or third attempt.

That is: P (Password is correct in the first attempt) + P(password is incorrect in the first attempt and correct in the second attempt) + P(password is incorrect in the first two attempts and correct in the third attempt)

= 1/100 + (1- 1/100)(1/99) + (1 - 1/100) (1 - 1/99) (1/98)
=3/100. **[2]**

(iii) P (Correct password is found on the 10th try)
= [P (Incorrect password in the first 9 attempts)][P(correct password on the 10th attempt)]
[1]
= [(1 – 1/100)^9][1/100] =0.009135. **[2]**

13.

i)

$$
\begin{aligned}
fx(x) &= \int_0^1 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) \mathrm{d}y \\
&= \left[\frac{3}{10}xy^3 + \frac{1}{5}y\right]_0^1 \\
&= \frac{3}{10}x + \frac{1}{5}
\end{aligned}
$$

$$
\begin{aligned}
f_Y(y) &= \int_0^2 \left(\frac{9}{10}xy^2 + \frac{1}{5}\right) \mathrm{d}x \\
&= \left[\frac{9}{20}x^2y^2 + \frac{1}{5}x\right]_0^2 \\
&= \frac{9}{5}y^2 + \frac{2}{5}
\end{aligned}
$$

ii)

$$
\begin{aligned}
E(X) &= \int_x x f_X(x)\,\mathrm{d}x \\
&= \int_0^2 x\left(\frac{3}{10}x + \frac{1}{5}\right) \mathrm{d}x \\
&= \int_0^2 \left(\frac{3}{10}x^2 + \frac{1}{5}x\right) \mathrm{d}x \\
&= \left[\frac{1}{10}x^3 + \frac{1}{10}x^2\right]_0^2 \\
&= \frac{8}{10} + \frac{4}{10} - 0 - 0 \\
&= \frac{6}{5} = 1.2
\end{aligned}
$$

$$
\begin{aligned}
E(Y) &= \int_y y f_Y(y)\,\mathrm{d}y \\
&= \int_0^1 y\left(\frac{9}{5}y^2 + \frac{2}{5}\right) \mathrm{d}y \\
&= \int_0^1 \left(\frac{9}{5}y^3 + \frac{2}{5}y\right) \mathrm{d}y \\
&= \left[\frac{9}{20}y^4 + \frac{1}{5}y^2\right]_0^1 \\
&= \frac{9}{20} + \frac{1}{5} \\
&= \frac{13}{20} = 0.65
\end{aligned}
$$

iii)

$$
\begin{aligned}
Cov(X,Y) &= \int_x \int_y xyf_{XY}(x,y)\,dy\,dx - E(X)E(Y) \\
&= \int_0^2 \int_0^1 xy\left(\frac{9}{10}xy^2 + \frac{1}{5}\right) dy\,dx - \left(\frac{6}{5}\right)\left(\frac{13}{20}\right) \\
&= \int_0^2 \int_0^1 \left(\frac{9}{10}x^2y^3 + \frac{1}{5}xy\right) dy\,dx - \frac{39}{50} \\
&= \int_0^2 \left[\frac{9}{40}x^2y^4 + \frac{1}{10}xy^2\right]_0^1 dx - \frac{39}{50} \\
&= \int_0^2 \left(\frac{9}{40}x^2 + \frac{1}{10}x\right) dx - \frac{39}{50} \\
&= \left[\frac{3}{40}x^3 + \frac{1}{20}x^2\right]_0^2 - \frac{39}{50} \\
&= \frac{3}{5} + \frac{1}{5} - \frac{39}{50} \\
&= \frac{1}{50} = 0.02
\end{aligned}
$$

14.

i) Given that X and *Y* are *iid* from exponential distribution with mean $\frac{1}{2}$.i.e. ~ *Exp*

$$
\begin{aligned}
&Fz(t) = P(Z \le t) = 1 - P(Min(X,Y) > t) \\
&= 1 - P(X > t \text{ and } Y > t) = 1 - P(X > t) * P(Y > y), \text{since } X \text{ and } Y \text{ are independent} \\
&= 1 - \{1 - P(X \le t)\} * \{1 - P(Y \le t)) = 1 - \{(1 - Fx(t)) * (1 - Fy(t))\} \\
&= 1 - \{1 - (1 - e^{-2t})\} * \{1 - (1 - e^{-2t})\} \qquad [as\ Fx(t) = Fy(t) = 1 - e^{-\lambda t}] \\
&= 1 - e^{-4t}
\end{aligned}
$$

ii) $1 - e^{-4t}$ is the CDF of an exponential distribution; hence the mean is: $\frac{1}{4}$.

15.

i) Given that $f_{X,Y}(x, y) = \frac{12}{5}(x^2y + xy)\,; 0 < x, y < 1.$

The marginal pdf of X : $h(x) = \int_0^1 \frac{12}{5}(x^2y + xy)\,dy = \left[\frac{12}{5}\left(\frac{1}{2}x^2y^2 + \frac{1}{2}xy^2\right)\right]_0^1$

$= \frac{12}{5}\left(\frac{1}{2}x^2 + \frac{1}{2}x\right)\ ;\quad 0 < x < 1.$

$= \frac{6}{5}\,(x^2 + x);\ \ 0 < x < 1$

The marginal pdf of Y : $g(y) = \int_0^1 \frac{12}{5}(x^2y + xy)\,dx = \left[\frac{12}{5}\left(\frac{1}{3}x^3y + \frac{1}{2}x^2y\right)\right]_0^1$

$= \frac{12}{5}\left(\frac{1}{3}y + \frac{1}{2}y\right);\ \ 0 < y < 1.$

$= 2y\ \ ;\quad 0 < y < 1$

(ii) Clearly $f_{X,Y}(x, y) = h(x)g(y)$. The random variables are statistically independent.

(iii) $E[X] = \int_0^1 \frac{12}{5}\left(\frac{1}{2}x^3 + \frac{1}{2}x^2\right)dx = \left[\frac{12}{5}\left(\frac{1}{8}x^4 + \frac{1}{6}x^3\right)\right]_{x=0}^1$

$= \frac{12}{5}\left(\frac{1}{8} + \frac{1}{6}\right) = 0.7$

$E[Y] = \int_0^1 \frac{12}{5}\left(\frac{1}{3}y^2 + \frac{1}{2}y^2\right)dy = \left[\frac{12}{5}\left(\frac{1}{9}y^3 + \frac{1}{6}y^3\right)\right]_{y=0}^1$

$= \frac{12}{5}\left(\frac{1}{9} + \frac{1}{6}\right) = 0.67.$

(iv) We know that $E(X/Y) = \int_0^1 x\,f(x|y)dx = \int_0^1 x\,\frac{f(xy)dx}{f(y)} = \int_0^1 \frac{\frac{12}{5}(x^3y + x^2y)}{2y}dx$

$= (6/5)\int_0^1 (x^3 + x^2)\,dx = (6/5)\left[\left(\frac{1}{4}x^4 + \frac{1}{3}x^3\right)\right]_{x=0}^1$

$= \frac{6}{5} \times \frac{7}{12} = \frac{7}{10}$

$$E(E(X/Y)) = \int_0^1 \left(\frac{7}{10}\right) f(x)dx = \int_0^1 \left(\frac{7}{10}\right) f(x)dx$$

$$= \int_0^1 \left(\frac{7}{10}\right) \frac{12}{5} [\frac{1}{2}(x^2 + x)]dx$$

$$= \frac{7}{10} \times \frac{12}{10} \left[\left(\frac{1}{3}x^3 + \frac{1}{2}x^2\right)\right]_{x=0}^{1} = \frac{7}{10} \times \frac{12}{10} \times \frac{5}{6} = 0.7 = E(X)$$

$$E(X^2/Y) = \int_0^1 x^2 f(x|y)dx = \int_0^1 x^2 \frac{f(xy)dx}{f(y)} = \int_0^1 \frac{\frac{12}{5}(x^4 y + x^3 y)}{2y} dx$$

$$= (6/5)\int_0^1 (x^4 + x^3)\, dx$$

$$= (6/5) \left[\left(\frac{1}{5}x^5 + \frac{1}{4}x^4\right)\right]_{x=0}^{1} = \frac{6}{5} \times \frac{9}{20} = \frac{54}{100}.$$

$$V(X/Y) = E(X^2/Y) - (E(X/Y))^2 = \frac{54}{100} - \left(\frac{7}{10}\right)^2 = \frac{1}{20}.$$

[If the candidate has answered using $E[X/Y] = E[X]$ and $V[X/Y] = V[X]$ *on computation of* $E[X^2]$ *and* $V[X]$ *full credit is to be given*

16.

i) By definition, the moment generating function of Gamma(α, λ):

$$M_X(t) = E(e^{tX}) = \int_0^\infty \left(e^{tx} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}\right) dx$$

$$= \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^\infty \left(x^{\alpha-1} e^{-(\lambda-t)x}\right) dx$$

$$M_X(t) = \frac{\lambda^\alpha}{(\lambda-t)^\alpha} \int_0^\infty \left(\frac{(\lambda-t)^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-(\lambda-t)x}\right) dx$$

The integrand is pdf of *Gamma* $(\alpha, \lambda - t)$ and the value of integral is 1.

Therefore, $M_X(t) = \left(\frac{\lambda}{\lambda-t}\right)^\alpha$, provided $t < \lambda$.

Dividing the numerator and denominator by λ gives:

$$M_X(t) = \left(\frac{1}{1-\frac{t}{\lambda}}\right)^\alpha = \left(1 - \frac{t}{\lambda}\right)^{-\alpha} \qquad t < \lambda$$

The cumulant generating function is:

$$C_X(t) = \log M_X(t) = -\alpha \log\left(1-\frac{t}{\lambda}\right)$$

ii) The coefficient of skewness = $\frac{Skew(X)}{Var(X)^{1.5}}$

We know that $Var(X) = C_X''(0)$ and $Skew(X) = C_X'''(0)$

$$C_X'(t) = \frac{\alpha}{\lambda}\left(1-\frac{t}{\lambda}\right)^{-1}$$

$$C_X''(t) = \frac{\alpha}{\lambda^2}\left(1-\frac{t}{\lambda}\right)^{-2} \Rightarrow Var(X) = C_X''(0) = \frac{\alpha}{\lambda^2}$$

$$C_X'''(t) = \frac{2\alpha}{\lambda^3}\left(1-\frac{t}{\lambda}\right)^{-3} \Rightarrow Skew(X) = C_X'''(0) = \frac{2\alpha}{\lambda^3}$$

Hence, the coefficient of skewness = $\frac{2\alpha/\lambda^3}{\left(\alpha/\lambda^2\right)^{1.5}} = \frac{2\alpha}{\alpha^{1.5}} = \frac{2}{\sqrt{\alpha}}$

17.

The moment generating function of $X_1 = M_{X_1}(t_1) = E(e^{t_1 X_1})$

$$= M_X(t_1, 0) = \frac{1}{3}\left(1 + e^{(t_1+2\times 0)} + e^{(2t_1+0)}\right)$$

$$= \frac{1}{3}\left(1 + e^{t_1} + e^{2t_1}\right)$$

The expected value of $X_1$is obtained bytaking first derivative of its MGF and evaluating at t_1=0.

Thus, $E(X_1) = M_{X_1}(0) = \frac{1}{3}\left(1 + e^0 + e^{0}\right) = 1$

Similarly using the mgf of X_2 , $E(X_2)$ is shown to be 1

$E(X_1X_2)$ is computed by taking the second cross-partial derivative of joint moment generating function evaluated at $(t_1,t_2) = (0,0)$:

$$\frac{\partial^2 M_{X_1,X_2}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial}{\partial t_1}\left(\frac{\partial}{\partial t_2}\left(\frac{1}{3}[1 + \exp(t_1 + 2t_2) + \exp(2t_1 + t_2)]\right)\right)$$

$$= \frac{\partial}{\partial t_1}\left(\frac{1}{3}[2\exp(t_1 + 2t_2) + \exp(2t_1 + t_2)]\right)$$

$$= \frac{1}{3}[2\exp(t_1 + 2t_2) + 2\exp(2t_1 + t_2)]$$

Thus, $E(X_1X_2) = \frac{4}{3}$

$Cov(X_1X_2) = E(X_1X_2) - E(X_1)E(X_2) = \frac{4}{3} - 1 \times 1 = \frac{1}{3}$ = 0.33

18.

i) The marginal *pdf* of Y:

i. $f_1(y) = \int_0^y 2\,dx = \begin{cases} 2y & 0<y<1 \\ 0, & otherwise. \end{cases}$

ii) The conditional *pdf* of X given Y: $f(x/y)$ is

i. $f(x/y) = \frac{f(x,y)}{f_1(y)} = \begin{cases} \frac{2}{2y}\ ; & 0<x<y;\ 0<y<1 \\ 0 & otherwise. \end{cases}$

iii) The conditional mean: $E(X/Y=2) = \int_0^y x\,\frac{1}{y}dx = \frac{y}{2},\ 0<y<1$

1. $= \frac{1}{4} = 0.25$ when $y = \frac{1}{2}$

iv) The conditional variance: $V(X/\ Y=y) = E(X^2/Y=y) - (E(X/Y=2))^2$

a. $E(X^2/Y=y) = \int_0^y x^2\frac{1}{y}\,dx = \frac{y^2}{3}\ ; 0<y<1$

1. $= \frac{1}{12} = 0.083$ when $y = \frac{1}{2}$.

Hence, $V(X/\ Y=\frac{1}{2}) = \frac{1}{12} - (\frac{1}{4})^2 = \frac{1}{48} = 0.0208$.

19.

The mean and variance of the Pareto distribution are given by:

$$E[X] = \frac{\lambda}{\alpha - 1} = \frac{4}{3-1} = 2$$

$$Var[X] = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = \frac{3(4)^2}{(3-1)^2(3-2)} = 12$$

From page 16 of the *Tables* $Var(Y) = Var[E(Y \mid X)] + E[Var(Y \mid X)]$:

$$Var(Y) = Var[2X + 5] + E[X^2 + 3]$$

$$Var(Y) = 4\,Var[X] + E[X^2] + 3$$

We know that $E[X^2] = Var[X] + (E[X])^2 = 12 + 2^2 = 16$

$Var(Y) = 4(12) + 16 + 3 = 67$

Hence, the standard deviations is $\sqrt{67} = 8.1854$

20.

(i) The Joint density of *(X,Y)*:

We know that $f(y|x) = \frac{f(x,y)}{f(x)}$.

Hence, $f(x,y) = f(y|x)f(x)$

$$= \frac{1}{x}(8x)$$

Thus, $f(x,y) = \begin{cases} 8 & \text{if } 0 < x < \frac{1}{2} \text{ and } 0 < y < x \\ 0 & \text{Otherwise.} \end{cases}$

(ii) The marginal density of *Y*:

$$f(y) = \int f(x,y)\,dx = \int_y^{0.5} 8\,dx = 4(1-2y) \quad \text{for } 0<y<\tfrac{1}{2}.$$

(iii) The Mean and Variance of *X* and *Y*:

$$E(Y)=\int_0^{0.5}(y)\,4\,(1-2y)\,dy = \frac{1}{2}-\frac{1}{3} = \frac{1}{6}.$$

$$V(Y) = E(Y^2)\text{-}(E(Y))^2.$$

$$E(Y^2)=\int_0^{0.5} y^2\,4(1-2y)\,dy = \frac{1}{6} - \frac{1}{8} = \frac{1}{24}.$$

$$V(Y)=\frac{1}{24} - \left(\frac{1}{6}\right)^2 = \frac{1}{72}.$$

21.

$$\begin{aligned} V(Y) &= E[V(Y|X)] + V[E(Y|X)] \\ &= E(X+1) + V(2X+3) \\ &= E(X) + 1 + 4\,V(X) \\ &= 5 + 1 + 4\,(5) \\ &= 26 \end{aligned}$$

22.

Revised Sample Mean = (213,200 – 11,000 – 48,000 + 27,500 + 31,500)/12 = (213,200/12) =17,767

Revised Σx^2 = 4,919,860,000 – 121,000,000 – 2,304,000,000 + 756,250,000 + 992,250,000 = 4,243,360,000

Therefore: Revised Sample Standard Deviation = $\{\frac{1}{11}\left(4{,}243{,}360{,}000 - \frac{213{,}200^2}{12}\right)\}^{1/2}$

$= \{\frac{1}{11}(455{,}506{,}666.7)\}^{1/2}$

$= \{41{,}409{,}696.97\}^{1/2}$

= 6,435.

Comment: There has been no change in the sample mean. However there has been a reduction in the Sample Standard Deviation from 10,144 to 6,435.

The mean remains unaltered as the total salary of the temporary employees who were being replaced is equal to the total salary of the permanent employees who replaced them. The reason for the reduction in standard deviation is that the salaries of the permanent employees who replaced temporary employees are closer to the sample mean.