IAQS | INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: P&S

Chapter: Unit 3

Category: Practice question

Chapter 7 - Generating Functions

1. **CT3 September 2010 Q8**

A certain type of claim amount (in units of £1,000) is modeled as an exponential random variable with parameter λ = 1.25. An analyst is interested in S, the total of 10 such independent claim amounts. In particular he wishes to calculate the probability that S exceeds £10,000.

(i) (a) Show, using moment generating functions, that:

(1) S has a gamma distribution, and

(2) 2.5 S has a chi-square 20 distribution.

(b) Use tables to calculate the required probability. [5]

[Ans – (i) b – 0.2014]

2. **CT3 September 2011 Q2**

The claims which arose in a sample of policies of a certain class gave the following frequency distribution for the number of claims per policy in the last year:

Number of claims (x)	0	1	2	3	4 or more
Number of policies (f)	15	20	10	5	0

Calculate the third order moment about the origin for these data. [3]

[Ans – 4.7]

3. **CT3 April 2014 Q4**

Let X be a random variable with probability density function:

$$f(x) = \{\tfrac{1}{2}e^{x} \quad x \leq 0 \; and \; \tfrac{1}{2}e^{-x} \quad x > 0$$

(i) Show that the moment generating function of X is given by:

$M_X(t) = (1 - t^2)^{-1}$ for $|t|<1$ [3]

(ii) Hence find the mean and the variance of X using the moment generating function in part (i). [3] [Total 6]

[Ans – (ii) – E(X) = 0, V(X) = 2]

4. **CT3 September 2018 Q2**

A random variable, X, has the probability generating function $G_X(t)$ where

$G_X(t) = 0.4096 + 0.4096t + 0.1528t^2 + 0.0256t^3 + 0.0016t^4$

(i) Determine the probability P(X = 3) using $G_X(t)$. [1]

You are now given that X follows a binomial distribution.

Unit 2

PRACTICE QUESTION

(ii) Determine the parameter values of the distribution of X. [3]
[Total 4]

[Ans –(i) - 0.0256, (ii) – 4, 0.2]

5. CS1A April 2021 Q3

Consider two random variables, X and Y, with a uniform distribution on the interval [0,1]; that is, X ~ U(0,1) and Y ~ U(0,1). Assume that X and Y are independent.

(i) Identify which one of the following options describes the moment generating function of X:

A. $\frac{1}{t}\left(e^{-t} - 1\right) \quad for\ t \neq 0$

B. $\frac{1}{t}\left(e^{t} - 1\right) \quad for\ t \neq 0$

C. $\frac{1}{t}\left(1 - e^{-t}\right) \quad for\ t \neq 0$

D. $\frac{1}{t}\left(1 - e^{t}\right) \quad for\ t \neq 0$ [2]

(ii) Derive the value of the moment generating function $M_X(t)$ of X at t = 0. [1]

An analyst argues that the sum of X and Y must have a uniform distribution on the interval [0,2] since both X and Y are uniformly distributed on [0,1].

(iii) Derive the moment generating function for the random variable Z with a U(0,2) distribution. [2]

(iv) Comment on the analyst's argument by determining if the random variable Z = X + Y has a uniform distribution on [0,2] using moment generating functions. [3]
[Total 8]

[Ans – (i) – B, (ii) – 1, (iii) – doesn't have uniform distribution]

Chapter 8 - Joint Distributions and Chapter 9 - Conditional Expectations

1. CT3 September 2011 Q5

Consider the random variable X taking the value X = 1 if a randomly selected person is a smoker, or X = 0 otherwise. The random variable Y describes the amount of physical exercise per week for this randomly selected person. It can take the values 0 (less than one hour of exercise per week), 1 (one to two hours) and 2 (more than two hours of exercise per week). The random variable R = $(3 - Y)^2(X + 1)$ is used as a risk index for a particular heart disease.

The joint distribution of X and Y is given by the joint probability function in the following table.

Unit 2

PRACTICE QUESTION

	Y		
X	0	1	2
0	0.2	0.3	0.25
1	0.1	0.1	0.05

(i) Calculate the probability that a randomly selected person does more than two hours of exercise per week. [1]
(ii) Decide whether X and Y are independent or not and justify your answer. [2]
(iii) Derive the probability function of R. [3]
(iv) Calculate the expectation of R. [2]
[Total 8]

[Ans – (i) – 0.3, (ii) – not independent, (iv) – 5.95]

2. CT3 September 2012 Q4

Consider a random variable U that has a uniform distribution on (0,1) and a random variable X that has a standard normal distribution. Assume that U and X are independent.
Determine an expression for the probability density function of the random variable Z = U + X in terms of the cumulative distribution function of X. [4]

3. CT3 April 2015 Q8

The random variables X and Y have a joint probability distribution with density function:

$$f_{xy}(x,y) = \begin{cases} 3x & 0 < y < x < 1 \\ 0 & otherwise \end{cases}$$

(i) Determine the marginal densities of X and Y. [4]
(ii) State, with reasons, whether X and Y are independent. [2]
(iii) Determine E[X] and E[Y]. [2]
[Total 8]

[Ans – (i) $f_X(x) = 3x^2$, $f_Y(y) = \frac{3}{2}(1-y^2)$, (ii) – not independent, (iii) E[X] =0.75, E[Y] = $\frac{3}{8}$]

4. CT3 April 2016 Q4

A manufacturing company is analysing its accident record. The accidents fall into two categories:

- Minor – dealt with by first aider. Average cost £50.
- Major – hospital visit required. Average cost £1,000.

The company has 1,000 employees, of which 180 are office staff and the rest work in the factory.
The analysis shows that 10% of employees have an accident each year and 20% of accidents are major. It is assumed that an employee has no more than one accident in a year.

Unit 2

PRACTICE QUESTION

(i) Determine the expected total cost of accidents in a year. [2]

On further analysis it is discovered that a member of office staff has half the probability of having an accident relative to those in the factory.

(ii) Show that the probability that a given member of office staff has an accident in a year is 0.0549. [3]

(iii) Determine the probability that a randomly chosen employee who has had an accident is office staff. [2]

[Total 7]

[Ans – (i) – 24000,(iii) – 0.099]

5. CT3 September 2016 Q6

Let X and Y be random variables with joint probability distribution:

$$f_{XY}(x,y) = \begin{cases} kx^2y^2 & 0 < x < y < 1 \\ 0 & otherwise \end{cases}$$

where k is a constant.

(i) Show that k = 18. [4]

(ii) Determine $f_Y(y)$, the marginal density function of Y. [2]

(iii) Determine P(X > 0.5|Y = 0.75). [3]

[Total 9]

[Ans - (ii) - $6y^5$, (iii) – 0.7037]

6. CT3 April 2018 Q9

The random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 24x^3y & 0 < x < y < 1 \\ 0 & otherwise \end{cases}$$

(i) (a) Show that the marginal pdf of X is

$$f_X(x) = 12x^3\left(1 - x^2\right), \quad 0 < x < 1$$

(b) Show that the marginal pdf of Y is

$f_Y(y) = 6y^5 \quad 0 < y < 1$ [2]

(ii) Determine the covariance cov (X, Y). [5]

(iii) Determine the conditional pdf $f_{X|Y}(x|y)$ together with the range of X for which it is defined. [2]

(iv) Determine the conditional probability $P(X > \frac{1}{3}|Y = \frac{1}{2})$. [2]

(v) Determine the conditional expectation $E\left(Y = \frac{1}{4}\right)$. [3]

(vi) Verify that E[E[X |Y]] = E[X] by evaluating each side of the equation. [3]

[Total 17]

Unit 2

PRACTICE QUESTION

[Ans – (ii) - $\frac{3}{245}$, (iii) - $4x^3y^{-4}$, (iv) - $\frac{65}{81}$, (v) - $\frac{1}{5}$, (vi) - $\frac{24}{35}$]

7. CS1A April 2019 Q6

Let X and Y be two continuous random variables.

(i) State the definition of independence of the random variables X and Y in terms of their joint probability density function. [2]

The joint probability density function of X and Y is given by:

$f_{X,Y}(x, y) = \{8xy \quad 0 < x < y < 1 \; 0 \quad otherwise$

(ii) (a) Determine the marginal density functions of X and Y. [2]

(b) State whether or not X and Y are independent based on your answer in part (ii)(a). [1]

(iii) Derive the conditional expectation E[X | Y = y]. [3]

[Total 8]

[Ans - (ii)(a) $f_X(x) = 4x(1 - x^2)$, $f_Y(y) = 4y^3$, (b) – not independent, (iii) – 2y/3]

8. CS1A April 2021 Q5

The joint probability density function of random variables X and Y is:

$$f(x, y) = \{ke^{-(x+2y)} \quad x > 0, y > 0 \; 0 \quad otherwise$$

[Hint: You may find it helpful to define the functions $g_X(x) = e^{-x}$ and $g_Y(y) = e^{-2y}$, using this notation in your answers.]

(i) Demonstrate that X and Y are independent. [1]

(ii) Verify that k = 2. [3]

(iii) Demonstrate that $f_Y(y)$, the marginal density function of Y, is: $2e^{-2y}$ for y > 0. [1]

(iv) Demonstrate that the conditional density function f(y|Y > 3) is:

$F(y|Y > 3) = 2e^{6-2y}$ for y > 3.

[Hint: Consider P(Y ≤ y|Y > 3).] [3]

(v) Identify which one of the following expressions is equal to the conditional expectation E[Y|Y > 3]:

A. $\int_0^\infty te^{-2t}dt + \int_0^\infty 3e^{-2t}dy$

Unit 2

PRACTICE QUESTION

B. $\int_0^\infty te^{-2t}dt + \int_0^\infty 6e^{-2t}dy$

C. $\int_0^\infty 2te^{-2t}dt + \int_0^\infty 3e^{-2t}dy$

D. $\int_0^\infty 2te^{-2t}dt + \int_0^\infty 6e^{-2t}dy$ [1]

(vi) Determine the value of the conditional expectation E[Y|Y > 3]. [2]

(vii) Identify which one of the following options is the conditional expectation E[Y^2|Y > 3]:

A. 12.5

B. 13.5

C. 14.5

D. 15.5. [2]

(viii) Determine the conditional variance Var[Y|Y > 3]. [1]

[Total 14]

[Ans – (v) – D, (vi) – 3.5, (vii) – D, (viii) – 0.25]

9. CS1A April 2022 Q3

Let X and Y be two continuous random variables jointly distributed with probability density function:

$$f_{XY}(x, y) = \{6e^{-(2x+3y)} \quad x, y \geq 0 \; 0 \quad otherwise$$

(i) Identify which one of the following options gives the correct expression for the marginal density function $f_X(x)$:

A. $f_X(x) = \{2e^{2x} \quad x \geq 0 \; 0 \quad otherwise$

B. $f_X(x) = \{e^{-2x} \quad x \geq 0 \; 0 \quad otherwise$

C. $f_X(x) = \{2e^{-x} \quad x \geq 0 \; 0 \quad otherwise$

D. $f_X(x) = \{2e^{-2x} \quad x \geq 0 \; 0 \quad otherwise$

(ii) Identify which one of the following options gives the correct expression for the marginal density function $f_Y(y)$:

A. $f_Y(y) = \{3e^{3y} \quad y \geq 0 \; 0 \quad otherwise$

B. $f_Y(y) = \{e^{3y} \quad y \geq 0 \; 0 \quad otherwise$

C. $f_Y(y) = \{3e^{-3y} \quad y \geq 0 \; 0 \quad otherwise$

D. $f_Y(y) = \{e^{-3y} \quad y \geq 0 \; 0 \quad otherwise$ [1]

(iii) Comment on whether X and *Y* are independent, by using your results in parts (i) and (ii). [1]

Unit 2

PRACTICE QUESTION

(iv) Calculate the conditional expectation E Y|X > 2.] [2]

(v) Identify which one of the following options gives the correct expression for P(X > Y):

A. 1/5

B. 3/5

C. 1/3

D. ½ [2]

[Total 7]

[Ans – (i) - D, (ii) – C, (iii) – independent, (iv) – 1/3, (v) – B]

10. CS1A September 2022 Q4

Consider two discrete random variables, X and Y, with the joint probability function given by:

	X = 1	X = 2	X = 3
Y = 0	0.3	0.1	0.3
Y = 1	0.05	0.2	0.05

(i) Verify that the table above specifies a joint distribution of two discrete random variables. [2]

(ii) Determine the expected value of X. [3]

(iii) Show that X and Y are not independent. [2]

[Total 7]

[Ans – (ii) – 2]

11. CT3 April 2010 Q6

Let X_1 ,X_2 ,.....X_n be a random sample of claim amounts which are modelled using a gamma distribution with known parameter α = 4 and unknown parameter λ.

(i) (a) Specify the distribution of $\sum_{i=1}^{n} X_i$

[Ans - (i) a - $\sum X_i$~ gamma(4n, λ)]

12. CT3 April 2011 Q4

Let N be the random variable that describes the number of claims that an insurer receives per month for one of its claim portfolios. We assume that N has a Poisson distribution with E[N] = 50. The amount X_i of each claim in the portfolio is normally distributed with mean μ = 1,000 and variance $\sigma^2 = 200^2$. The total amount of all claims received during one month is

$$S = \sum_{i=1}^{n} X_i$$

with S = 0 for N = 0. We assume that N, X_1, X_2, ... are all independent of each other.

(i) Specify the type of the distribution of S. [1]

(ii) Calculate the mean and standard deviation of S. [3]

[Total 4]

Unit 2

PRACTICE QUESTION

[Ans – (i) – Compound poisson distribution, (ii) – E[S] = 50000, SD[S] = 7211.10]

13. CT3 April 2011 Q5

Let X_1, X_2, X_3, X_4, and X_5 be independent random variables, such that X_i ~ gamma with parameters i and λ for i = 1, 2, 3, 4, 5. Let $S = 2\lambda \sum_{i=1}^{5} X_i$

(i) Derive the mean and variance of S using standard results for the mean and variance of linear combinations of random variables. [3]

(ii) Show that S has a chi-square distribution using moment generating functions and state the degrees of freedom of this distribution. [4]

(iii) Verify the values found in part (i) using the results of part (ii). [1]

[Total 8]

[Ans – (i) – E[S] = 30, V[S] = 60, (ii) – df = 30]

14. CT3 September 2011 Q4

The random variables X and Y are related as follows:

X conditional on Y = y has a N(2y, y^2) distribution.

Y has a N (200, 100) distribution.

Derive the unconditional variance of X, V[X]. [3]

[Ans – 40500]

15. CT3 April 2012 Q10

In a portfolio of car insurance policies, the number of accident-related claims, N, made by a policyholder in a year has the following distribution:

No of claims (n)	0	1	2
Probability	0.4	0.4	0.2

The number of cars, X, involved in each accident that results in a claim is distributed as follows:

No of cars (x)	1	2
Probability	0.7	0.3

It can be assumed that the occurrence of a claim and the number of cars involved in the accident are independent. Furthermore, claims made by a policyholder in any year are also independent of each other. Let S be the total number of cars involved in accidents related to such claims by a policyholder in a year.

(i) (a) Determine the probability function of S.

(b) Hence find E(S). [4]

The expectation E(S) can also be calculated using the formula

$$E(S) = \sum_{n=0}^{2} E(N = n)Pr(N = n)$$

(ii) (a) Find E(S|N=n) for n = 0,1, 2.

(b) Hence calculate E(S). [4]

[Total 8]

[Ans - (i) a - P(S = 0) = 0.4, P(S = 1) = 0.28, P(S = 2) = 0.218, P(S = 3) = 0.084, P(S = 4) = 0.018, b – E[S] = 1.04, (ii)a - E(S|N = 0) = 0, E(S|N = 1) = 1.3, E(S|N = 2) = 2.6, b- E[S] = 1.04]

16. CT3 September 2012 Q8

The random variable S is given as S = $Y_1 + Y_2 + \ldots + Y_N$ (with S = 0 if N = 0) where the random variables Yi are identically and independently distributed according to a lognormal distribution with parameters μ = 0.5 and $\sigma^2 = 0.1$. N is also a random variable which is independent of Y_i , and its distribution given below.

N	0	1	2	3	4
Pr(N=n)	0.1	0.3	0.3	0.2	0.1

Calculate the mean and the variance of the random variable S. [7]

{Ans – E[S] = 3.293, V[S] = 4.476]

17. CT3 April 2014 Q7

Let X and Y be two continuous random variables.

(i) Prove that E[E[Y|X]] = E[Y] [3]

Suppose the number of claims, N, on a policy follows a Poisson distribution with mean μ, and the amount of the i^{th} claim, X_i, follows a Gamma distribution with parameters α and λ. Let S denote the total value of claims on a policy in a given year.

(ii) Derive the mean of S using the result in part (i). [2]

Suppose μ = 0.15, α = 100, λ= 0.1

(iii) Calculate the variance of S. [3]

[Total 8]

[Ans – (ii) – μα/ λ, (iii) – 151,500]

18. CT3 September 2014 Q3

Let N be a random variable describing the number of withdrawals from a bank branch each day. It is assumed that N is Poisson distributed with mean μ. Let Xi, the random variable describing the amount of each withdrawal, be exponentially distributed with mean 1/λ. All X_i are independent and identically distributed. Let S denote the total amount withdrawn from that branch in a day i.e.

$$S = \sum_{i=1}^{n} X_i$$

with S = 0 if N = 0.

(i) Derive the moment generating function of S. [4]

(ii) Calculate the mean and variance of S if μ = 100 and λ = 0.025. [3]

[Total 7]

[Ans – (i) – $M_S(t) = exp[\mu\{(1-\frac{t}{\lambda})^{-1} - 1\}]$, (ii) - E[S] = 4000, V[S] = 320000]

19. CT3 September 2015 Q7

X and Y are discrete random variables with joint distribution given below.

	Y = -1	Y = 0	Y = 1
X = 1	0	1/4	0
X = 0	1/4	1/4	¼

(i) Determine the conditional expectation E[Y|X = 1]. [1]

(ii) Determine the conditional expectation E[X|Y = y] for each value of y. [3]

(iii) Determine the expected value of X based on your conditional expectation results from part (ii). [2]

[Total 6]

[Ans – (i) – 0, (ii) – 0,1/2, 0, (iii) – ¼]

20. CT3 September 2016 Q9 (only part ii)

A statistical model is used to describe the total loss, S (in pounds), experienced in a certain portfolio of an insurance company over a period of one year. The total loss is given by:

$S = X_1 + X_2 + \ldots\ldots + X_N$

where X_i gives the size of the loss from claim i =1,…, N. N is a random variable representing the number of claims per year and follows a Poisson distribution. The X_i s are independent, identically distributed according to a gamma distribution with parameters α and λ, and are also independent of N.

Data from previous years show that the average number of claims per year was 14, while the average size of claims was £500 and their standard deviation was £150.

Unit 2

PRACTICE QUESTION

(i) Estimate the parameters α and λ using the method of moments. [4]

(ii) Estimate the mean and the variance of the total loss S using the information from the data above. [3]

[Ans – (ii) – 7000, 3815000]

21.CT3 September 2017 Q7

The annual number of claims an insurance company incurs, N, is believed to follow a Poisson distribution with mean l. The value of each claim Xi, i = 1, 2, … follows a known distribution with mean m and variance s2. The value of each claim is independent of the value of any other claim and of the number of claims. Let $S = X_1 + X_2 + \ldots + X_N$ denote the total claims in any given year.

(i) Write down an expression for the moment generating function of S in terms of the moment generating function of Xi. [1]

(ii) Derive formulae for the mean and variance of S using your answer to part (i). [5]

[Total 6]

[Ans – (i) - $M_S(y) = exp\{\lambda\left(M_X(y) - 1\right)\}$, (ii) – $E[S] = \lambda\mu$, $V[S] = \lambda(\sigma^2 + \mu^2)$]

22. CS1A September 2019 Q4

X and Y are discrete random variables with joint distribution as follows:

	X=0	X=1	X=3
Y=-1	0.08	0.03	0
Y=0	0.03	0.12	0.20
Y=3	0.11	0.11	0.06
Y=4.5	0.04	0.20	0.02

(i) Calculate:

(a) E(Y | X = 1) (b) Var(X | Y = 3). [5]

(ii) Calculate the probability functions of the marginal distributions for X and Y. [2]

(iii) Determine whether X and Y are independent. [2]

[Total 9]

[Ans – (i) (a) – 2.6087, (b) – 1.2487, (ii) - $P(X = 0)$= 0.26,$P(X = 1)$= 0.46,$P(X=3)$ = 0.28, $P(Y = -1)$= 0.11,$P(Y = 0)$= 0.35,$P(Y = 3)$= 0.28, (iii) – not independent]

23. CS1A September 2020 Q1

Let $X_1, X_2, \ldots, X_{81}$ be independent and identically distributed continuous random variables, each with expected value $\mu = E(X_i) = 5$, and variance $\sigma^2 = V(X_i) = 4$.

Unit 2

PRACTICE QUESTION

(i) Determine the sampling distribution of the statistic $T = \sum_{i=1}^{81} X_i$. [2]

(ii) Calculate the probability P(T > 369) using your answer to part (i). [2]

[Total 4]

[Ans – (i) – T ~ N(405,324), (ii) – 0.97725]

24. CS1A April 2021 Q2

Consider two random variables, X and Y. The conditional expectation and conditional variance of Y given X are denoted by the two random variables U and V, respectively; that is, U = E[Y|X] and V = Var[Y|X]. Assume that Y is Normally distributed with expectation 5 and variance 4. Also assume that the expectation of V is 2.

(i) Calculate the expected value of U. [1]

(ii) Calculate the variance of U. [2]

[Total 3]

[Ans – (i) – 5, (ii) – 2]

25. CS1A September 2021 Q7

Let X_i , i = 1, 2, …, n be independent random variables, each following an exponential distribution with parameter b. We consider the random variable $Y = \sum_{i=1}^{n} X_i$

(i) Justify why $M_Y t$, the moment generating function (MGF) of variable Y, is given by $M_Y t = (1 - t/b)^{-n}$ [2]

Let Z be a random variable such that the MGF of Z is $M_z t = \sqrt{M_Y t}$.

(ii) Determine the value of b for which Z follows a chi-square distribution, specifying the degrees of freedom of the chi-square distribution. [3]

[Total 5]

[Ans – (ii) – 0.5]

Unit 2

PRACTICE QUESTION