



INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Subject: Probability and
Statistics

Chapter: Unit 2

Category: Practice question

Chapter 7 - Generating Functions

1. CT3 September 2010 Q8

A certain type of claim amount (in units of £1,000) is modeled as an exponential random variable with parameter $\lambda = 1.25$. An analyst is interested in S , the total of 10 such independent claim amounts. In particular he wishes to calculate the probability that S exceeds £10,000.

(i) (a) Show, using moment generating functions, that:

(1) S has a gamma distribution, and

(2) $2.5 S$ has a chi-square 20 distribution.

(b) Use tables to calculate the required probability. [5]

[Ans – (i) b – 0.2014]

2. CT3 September 2011 Q2

The claims which arose in a sample of policies of a certain class gave the following frequency distribution for the number of claims per policy in the last year:

Number of claims (x)	0	1	2	3	4 or more
Number of policies (f)	15	20	10	5	0

Calculate the third order moment about the origin for these data. [3]

[Ans – 4.7]

3. CT3 April 2014 Q4

Let X be a random variable with probability density function:

$$f(x) = \begin{cases} \frac{1}{2}e^x & x \leq 0 \\ \frac{1}{2}e^{-x} & x > 0 \end{cases}$$

(i) Show that the moment generating function of X is given by:

$$M_X(t) = (1 - t^2)^{-1} \quad \text{for } |t| < 1 \quad [3]$$

(ii) Hence find the mean and the variance of X using the moment generating function in part (i). [3]

[Ans – (ii) – $E(X) = 0$, $V(X) = 2$]

4. CT3 September 2018 Q2

A random variable, X , has the probability generating function $G_X(t)$ where

$$G_X(t) = 0.4096 + 0.4096t + 0.1528t^2 + 0.0256t^3 + 0.0016t^4$$

(i) Determine the probability $P(X = 3)$ using $G_X(t)$. [1]

You are now given that X follows a binomial distribution.

(ii) Determine the parameter values of the distribution of X . [3]

[Total 4]

[Ans –(i) - 0.0256, (ii) – 4, 0.2]

5. CS1A April 2021 Q3

Consider two random variables, X and Y , with a uniform distribution on the interval $[0,1]$; that is, $X \sim U(0,1)$ and $Y \sim U(0,1)$. Assume that X and Y are independent.

(i) Identify which one of the following options describes the moment generating function of X :

A. $\frac{1}{t}(e^{-t} - 1)$ for $t \neq 0$

B. $\frac{1}{t}(e^t - 1)$ for $t \neq 0$

C. $\frac{1}{t}(1 - e^{-t})$ for $t \neq 0$

D. $\frac{1}{t}(1 - e^t)$ for $t \neq 0$ [2]

(ii) Derive the value of the moment generating function $M_X(t)$ of X at $t = 0$. [1]

An analyst argues that the sum of X and Y must have a uniform distribution on the interval $[0,2]$ since both X and Y are uniformly distributed on $[0,1]$.

(iii) Derive the moment generating function for the random variable Z with a $U(0,2)$ distribution. [2]

(iv) Comment on the analyst's argument by determining if the random variable $Z = X + Y$ has a uniform distribution on $[0,2]$ using moment generating functions. [3]

[Total 8]

[Ans – (i) – B, (ii) – 1, (iii) – doesn't have uniform distribution]

Chapter 8 - Joint Distributions and Chapter 9 - Conditional Expectations

1. CT3 September 2011 Q5

Consider the random variable X taking the value $X = 1$ if a randomly selected person is a smoker, or $X = 0$ otherwise. The random variable Y describes the amount of physical exercise per week for this randomly selected person. It can take the values 0 (less than one hour of exercise per week), 1 (one to two hours) and 2 (more than two hours of exercise per week). The random variable $R = (3 - Y)^2(X + 1)$ is used as a risk index for a particular heart disease.

The joint distribution of X and Y is given by the joint probability function in the following table.

	Y		
X	0	1	2
0	0.2	0.3	0.25
1	0.1	0.1	0.05

- Calculate the probability that a randomly selected person does more than two hours of exercise per week. [1]
- Decide whether X and Y are independent or not and justify your answer. [2]
- Derive the probability function of R . [3]
- Calculate the expectation of R . [2]

[Total 8]

[Ans – (i) – 0.3, (ii) – not independent, (iv) – 5.95]

2. CT3 September 2012 Q4

Consider a random variable U that has a uniform distribution on $(0,1)$ and a random variable X that has a standard normal distribution. Assume that U and X are independent.

Determine an expression for the probability density function of the random variable $Z = U + X$ in terms of the cumulative distribution function of X . [4]

3. CT3 April 2015 Q8

The random variables X and Y have a joint probability distribution with density function:

$$f_{xy}(x, y) = \begin{cases} 3x & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

- (i) Determine the marginal densities of X and Y. [4]
- (ii) State, with reasons, whether X and Y are independent. [2]
- (iii) Determine E[X] and E[Y]. [2]

[Ans – (i) $f_X(x) = 3x^2, f_Y(y) = \frac{3}{2}(1 - y^2)$, (ii) – not independent, (iii) $E[X] = 0.75, E[Y] = \frac{3}{8}$]

4. CT3 April 2016 Q4

A manufacturing company is analysing its accident record. The accidents fall into two categories:

- Minor – dealt with by first aider. Average cost £50.
- Major – hospital visit required. Average cost £1,000.

The company has 1,000 employees, of which 180 are office staff and the rest work in the factory.

The analysis shows that 10% of employees have an accident each year and 20% of accidents are major. It is assumed that an employee has no more than one accident in a year.

- (i) Determine the expected total cost of accidents in a year. [2]

On further analysis it is discovered that a member of office staff has half the probability of having an accident relative to those in the factory.

- (ii) Show that the probability that a given member of office staff has an accident in a year is 0.0549. [3]
- (iii) Determine the probability that a randomly chosen employee who has had an accident is office staff. [2]

[Ans – (i) – 24000, (iii) – 0.099]

5. CT3 September 2016 Q6

Let X and Y be random variables with joint probability distribution:

$$f_{XY}(x, y) = \begin{cases} kx^2y^2 & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

where k is a constant.

- (i) Show that $k = 18$. [4]
- (ii) Determine $f_Y(y)$, the marginal density function of Y . [2]
- (iii) Determine $P(X > 0.5 | Y = 0.75)$. [3]

[Ans - (ii) - $6y^5$, (iii) - 0.7037]

6. CT3 April 2018 Q9

The random variables X and Y have joint probability density function (pdf)

$$f_{X,Y}(x,y) = \begin{cases} 24x^3y & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

- (i) (a) Show that the marginal pdf of X is

$$f_X(x) = 12x^3(1 - x^2), \quad 0 < x < 1$$

- (b) Show that the marginal pdf of Y is

$$f_Y(y) = 6y^5 \quad 0 < y < 1 \quad [2]$$

- (ii) Determine the covariance $\text{cov}(X, Y)$. [5]
 - (iii) Determine the conditional pdf $f_{X|Y}(x|y)$ together with the range of X for which it is defined. [2]
 - (iv) Determine the conditional probability $P(X > \frac{1}{3} | Y = \frac{1}{2})$. [2]
 - (v) Determine the conditional expectation $E(Y = \frac{1}{4})$. [3]
 - (vi) Verify that $E[E[X | Y]] = E[X]$ by evaluating each side of the equation. [3]
- [Total 17]

[Ans - (ii) - $\frac{3}{245}$, (iii) - $4x^3y^{-4}$, (iv) - $\frac{65}{81}$, (v) - $\frac{1}{5}$, (vi) - $\frac{24}{35}$]

7. CS1A April 2019 Q6

Let X and Y be two continuous random variables.

- (i) State the definition of independence of the random variables X and Y in terms of their joint probability density function. [2]

The joint probability density function of X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} 8xy & 0 < x < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

- (ii) (a) Determine the marginal density functions of X and Y . [2]

(b) State whether or not X and Y are independent based on your answer in part (ii)(a). [1]

(iii) Derive the conditional expectation $E[X | Y = y]$. [3]

[Total 8]

[Ans - (ii)(a) $f_X(x) = 4x(1 - x^2)$, $f_Y(y) = 4y^3$, (b) – not independent, (iii) – $2y/3$]

8. CS1A April 2021 Q5

The joint probability density function of random variables X and Y is:

$$f(x, y) = \begin{cases} ke^{-(x+2y)} & x > 0, y > 0 \\ 0 & \text{otherwise} \end{cases}$$

[Hint: You may find it helpful to define the functions $g_X(x) = e^{-x}$ and $g_Y(y) = e^{-2y}$, using this notation in your answers.]

(i) Demonstrate that X and Y are independent. [1]

(ii) Verify that $k = 2$. [3]

(iii) Demonstrate that $f_Y(y)$, the marginal density function of Y , is: $2e^{-2y}$ for $y > 0$. [1]

(iv) Demonstrate that the conditional density function $f(y | Y > 3)$ is:

$$f(y | Y > 3) = 2e^{6-2y} \text{ for } y > 3.$$

[Hint: Consider $P(Y \leq y | Y > 3)$.] [3]

(v) Identify which one of the following expressions is equal to the conditional expectation $E[Y | Y > 3]$:

- A. $\int_0^\infty te^{-2t} dt + \int_0^\infty 3e^{-2t} dy$
- B. $\int_0^\infty te^{-2t} dt + \int_0^\infty 6e^{-2t} dy$
- C. $\int_0^\infty 2te^{-2t} dt + \int_0^\infty 3e^{-2t} dy$
- D. $\int_0^\infty 2te^{-2t} dt + \int_0^\infty 6e^{-2t} dy$ [1]

(vi) Determine the value of the conditional expectation $E[Y | Y > 3]$. [2]

(vii) Identify which one of the following options is the conditional expectation $E[Y^2 | Y > 3]$:

- A. 12.5

- B. 13.5
C. 14.5
D. 15.5. [2]

(viii) Determine the conditional variance $\text{Var}[Y | Y > 3]$. [1]

[Total 14]

[Ans – (v) – D, (vi) – 3.5, (vii) – D, (viii) – 0.25]

9. CS1A April 2022 Q3

Let X and Y be two continuous random variables jointly distributed with probability density function:

$$f_{XY}(x, y) = \begin{cases} 6e^{-(2x+3y)} & x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

(i) Identify which one of the following options gives the correct expression for the marginal density function $f_X(x)$:

- A. $f_X(x) = \begin{cases} 2e^{2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 B. $f_X(x) = \begin{cases} e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 C. $f_X(x) = \begin{cases} 2e^{-x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 D. $f_X(x) = \begin{cases} 2e^{-2x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$

(ii) Identify which one of the following options gives the correct expression for the marginal density function $f_Y(y)$:

- A. $f_Y(y) = \begin{cases} 3e^{3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 B. $f_Y(y) = \begin{cases} e^{3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 C. $f_Y(y) = \begin{cases} 3e^{-3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$
 D. $f_Y(y) = \begin{cases} e^{-3y} & y \geq 0 \\ 0 & \text{otherwise} \end{cases}$ [1]

(iii) Comment on whether X and Y are independent, by using your results in parts (i) and (ii). [1]

(iv) Calculate the conditional expectation $E[Y | X > 2]$. [2]

(v) Identify which one of the following options gives the correct expression for $P(X > Y)$:

- A. $1/5$
B. $3/5$
C. $1/3$
D. $1/2$ [2]

[Ans – (i) – D, (ii) – C, (iii) – independent, (iv) – $1/3$, (v) – B]

10. CS1A September 2022 Q4

Consider two discrete random variables, X and Y , with the joint probability function given by:

	$X = 1$	$X = 2$	$X = 3$
$Y = 0$	0.3	0.1	0.3
$Y = 1$	0.05	0.2	0.05

- (i) Verify that the table above specifies a joint distribution of two discrete random variables. [2]
- (ii) Determine the expected value of X . [3]
- (iii) Show that X and Y are not independent. [2]

[Ans – (ii) – 2]

11. CT3 April 2010 Q6

Let $X_1, X_2, \dots, X_n$ be a random sample of claim amounts which are modelled using a gamma distribution with known parameter $\alpha = 4$ and unknown parameter λ .

- (i) (a) Specify the distribution of $\sum_{i=1}^n X_i$

[Ans - (i) a - $\sum_{i=1}^n X_i \sim \text{gamma}(4n, \lambda)$]

12. CT3 April 2011 Q4

Let N be the random variable that describes the number of claims that an insurer receives per month for one of its claim portfolios. We assume that N has a Poisson distribution with $E[N] = 50$. The amount X_i of each claim in the portfolio is normally distributed with mean $\mu = 1,000$ and variance $\sigma^2 = 200^2$. The total amount of all claims received during one month is

$$S = \sum_{i=1}^N X_i$$

with $S = 0$ for $N = 0$. We assume that $N, X_1, X_2, \dots$ are all independent of each other.

- (i) Specify the type of the distribution of S . [1]
- (ii) Calculate the mean and standard deviation of S . [3]

[Total 4]

[Ans – (i) – Compound poisson distribution, (ii) – $E[S] = 50000$, $SD[S] = 7211.10$]

13. CT3 April 2011 Q5

Let X_1, X_2, X_3, X_4 , and X_5 be independent random variables, such that $X_i \sim \text{gamma}$ with parameters i and λ for $i = 1, 2, 3, 4, 5$. Let $S = 2\lambda \sum_{i=1}^5 X_i$

- (i) Derive the mean and variance of S using standard results for the mean and variance of linear combinations of random variables. [3]
 (ii) Show that S has a chi-square distribution using moment generating functions and state the degrees of freedom of this distribution. [4]
 (iii) Verify the values found in part (i) using the results of part (ii). [1]

[Total 8]

[Ans – (i) – $E[S] = 30$, $V[S] = 60$, (ii) – $df = 30$]

14. CT3 September 2011 Q4

The random variables X and Y are related as follows:

X conditional on $Y = y$ has a $N(2y, y^2)$ distribution.

Y has a $N(200, 100)$ distribution.

Derive the unconditional variance of X , $V[X]$. [3]

[Ans – 40500]

15. CT3 April 2012 Q10

In a portfolio of car insurance policies, the number of accident-related claims, N , made by a policyholder in a year has the following distribution:

No of claims (n)	0	1	2
Probability	0.4	0.4	0.2

The number of cars, X , involved in each accident that results in a claim is distributed as follows:

No of cars (x)	1	2
Probability	0.7	0.3

It can be assumed that the occurrence of a claim and the number of cars involved in the accident are independent. Furthermore, claims made by a policyholder in any year are also independent of each other. Let S be the total number of cars involved in accidents related to such claims by a policyholder in a year.

- (i) (a) Determine the probability function of S.
 (b) Hence find E(S). [4]

The expectation E(S) can also be calculated using the formula

$$E(S) = \sum_{n=0}^2 E(N = n)Pr(N = n)$$

- (ii) (a) Find E(S | N=n) for n = 0, 1, 2.
 (b) Hence calculate E(S).

[Ans - (i) a - P(S = 0) = 0.4, P(S = 1) = 0.28, P(S = 2) = 0.218, P(S = 3) = 0.084, P(S = 4) = 0.018, b - E[S] = 1.04, (ii) a - E(S | N = 0) = 0, E(S | N = 1) = 1.3, E(S | N = 2) = 2.6, b - E[S] = 1.04]

16. CT3 September 2012 Q8

The random variable S is given as $S = Y_1 + Y_2 + \dots + Y_N$ (with $S = 0$ if $N = 0$) where the random variables Y_i are identically and independently distributed according to a lognormal distribution with parameters $\mu = 0.5$ and $\sigma^2 = 0.1$. N is also a random variable which is independent of Y_i , and its distribution given below.

N	0	1	2	3	4
Pr(N=n)	0.1	0.3	0.3	0.2	0.1

Calculate the mean and the variance of the random variable S. [7]

{Ans - E[S] = 3.293, V[S] = 4.476}

17. CT3 April 2014 Q7

Let X and Y be two continuous random variables.

- (i) Prove that $E[E[Y | X]] = E[Y]$ [3]

Suppose the number of claims, N, on a policy follows a Poisson distribution with mean μ , and the amount of the i^{th} claim, X_i , follows a Gamma distribution with parameters α and λ . Let S denote the total value of claims on a policy in a given year.

(ii) Derive the mean of S using the result in part (i). [2]

Suppose $\mu = 0.15$, $\alpha = 100$, $\lambda = 0.1$

(iii) Calculate the variance of S . [3]

[Ans – (ii) – $\mu\alpha / \lambda$, (iii) – 151,500]

18. CT3 September 2014 Q3

Let N be a random variable describing the number of withdrawals from a bank branch each day. It is assumed that N is Poisson distributed with mean μ . Let X_i , the random variable describing the amount of each withdrawal, be exponentially distributed with mean $1/\lambda$. All X_i are independent and identically distributed. Let S denote the total amount withdrawn from that branch in a day i.e.

$$S = \sum_{i=1}^N X_i$$

with $S = 0$ if $N = 0$.

(i) Derive the moment generating function of S . [4]

(ii) Calculate the mean and variance of S if $\mu = 100$ and $\lambda = 0.025$. [3]

[Total 7]

[Ans – (i) – $M_S(t) = \exp[\mu\{(1 - \frac{t}{\lambda})^{-1} - 1\}]$, (ii) – $E[S] = 4000$, $V[S] = 320000$]

19. CT3 September 2015 Q7

X and Y are discrete random variables with joint distribution given below.

	$Y = -1$	$Y = 0$	$Y = 1$
$X = 1$	0	$1/4$	0
$X = 0$	$1/4$	$1/4$	$1/4$

(i) Determine the conditional expectation $E[Y|X = 1]$. [1]

(ii) Determine the conditional expectation $E[X|Y = y]$ for each value of y . [3]

(iii) Determine the expected value of X based on your conditional expectation results from part (ii). [2]

[Total 6]

[Ans – (i) – 0, (ii) – 0, $1/2$, 0, (iii) – $1/4$]

20. CT3 September 2016 Q9 (only part ii)

A statistical model is used to describe the total loss, S (in pounds), experienced in a certain portfolio of an insurance company over a period of one year. The total loss is given by:

$$S = X_1 + X_2 + \dots + X_N$$

where X_i gives the size of the loss from claim $i = 1, \dots, N$. N is a random variable representing the number of claims per year and follows a Poisson distribution. The X_i s are independent, identically distributed according to a gamma distribution with parameters α and λ , and are also independent of N .

Data from previous years show that the average number of claims per year was 14, while the average size of claims was £500 and their standard deviation was £150.

(i) Estimate the parameters α and λ using the method of moments. [4]

(ii) Estimate the mean and the variance of the total loss S using the information from the data above. [3]

[Ans – (ii) – 7000, 3815000]

21. CT3 September 2017 Q7

The annual number of claims an insurance company incurs, N , is believed to follow a Poisson distribution with mean 1. The value of each claim X_i , $i = 1, 2, \dots$ follows a known distribution with mean m and variance s^2 . The value of each claim is independent of the value of any other claim and of the number of claims. Let $S = X_1 + X_2 + \dots + X_N$ denote the total claims in any given year.

(i) Write down an expression for the moment generating function of S in terms of the moment generating function of X_i . [1]

(ii) Derive formulae for the mean and variance of S using your answer to part (i). [5]
[Total 6]

[Ans – (i) – $M_S(y) = \exp \{ \lambda (M_X(y) - 1) \}$, (ii) – $E[S] = \lambda \mu$, $V[S] = \lambda (\sigma^2 + \mu^2)$]

22. CS1A September 2019 Q4

X and Y are discrete random variables with joint distribution as follows:

	X=0	X=1	X=3
Y=-1	0.08	0.03	0
Y=0	0.03	0.12	0.20
Y=3	0.11	0.11	0.06
Y=4.5	0.04	0.20	0.02

(i) Calculate:

(a) $E(Y | X = 1)$ (b) $\text{Var}(X | Y = 3)$. [5]

(ii) Calculate the probability functions of the marginal distributions for X and Y. [2]

(iii) Determine whether X and Y are independent. [2]

[Total 9]

[Ans – (i) (a) – 2.6087, (b) – 1.2487, (ii) - $P(X = 0) = 0.26, P(X = 1) = 0.46, P(X=3) = 0.28$,
 $P(Y = -1) = 0.11, P(Y = 0) = 0.35, P(Y = 3) = 0.28$, (iii) – not independent]

23. CS1A September 2020 Q1

Let $X_1, X_2, \dots, X_{81}$ be independent and identically distributed continuous random variables, each with expected value $\mu = E(X_i) = 5$, and variance $\sigma^2 = V(X_i) = 4$.

(i) Determine the sampling distribution of the statistic $T = \sum_{i=1}^{81} X_i$. [2]

(ii) Calculate the probability $P(T > 369)$ using your answer to part (i). [2]

[Total 4]

[Ans – (i) – $T \sim N(405, 324)$, (ii) – 0.97725]

24. CS1A April 2021 Q2

Consider two random variables, X and Y. The conditional expectation and conditional variance of Y given X are denoted by the two random variables U and V, respectively; that is, $U = E[Y | X]$ and $V = \text{Var}[Y | X]$. Assume that Y is Normally distributed with expectation 5 and variance 4. Also assume that the expectation of V is 2.

(i) Calculate the expected value of U. [1]

(ii) Calculate the variance of U. [2]

[Total 3]

[Ans – (i) – 5, (ii) – 2]

25. CS1A September 2021 Q7

Let X_i , $i = 1, 2, \dots, n$ be independent random variables, each following an exponential distribution with parameter b . We consider the random variable $Y = \sum_{i=1}^n X_i$

(i) Justify why M_{Yt} , the moment generating function (MGF) of variable Y , is given by

$$M_{Yt} = (1 - t/b)^{-n} \quad [2]$$

Let Z be a random variable such that the MGF of Z is $M_{Zt} = \sqrt{M_{Yt}}$.

(ii) Determine the value of b for which Z follows a chi-square distribution, specifying the degrees of freedom of the chi-square distribution. [3]

[Total 5]

[Ans – (ii) – 0.5]