INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Probability and Statistics

Chapter: Continuous Distributions

Category: Practice question – Extras

Continuous Uniform Distribution:

1. *A random variable X is uniformly distributed between 32 and 42. What is the probability that X will be between 32 and 40?*

Ans: 80%

f(x) = $\frac{1}{b-a} = \frac{1}{10}$

P(32 < X < 40) = $= \frac{40-32}{10} = \frac{8}{10}$

2. X follows uniform distribution in the span (-4, 1). What is the mean and variance of x.

a = -4 & b =1

Calculation:

Mean:

$\mu = \frac{-4+1}{2} = \frac{-3}{2} = -1.5$

Variance:

$\sigma^2 = \frac{(1-(-4))^2}{12} = \frac{(5)^2}{12} = \frac{25}{12}$

3. A student tries to breaks his pencil of unit length two pieces uniformly. The expected length of the shorter piece of pencil is :

Ans: 1/4

Let x be the length of the shorter piece of pencil. Hence it varies between $\left[0-\frac{1}{2}\right]$.

By using uniform distribution, p.d.f of the given condition:

$$f(x)=\begin{cases}\frac{1}{b-a} & ; \quad a<x\leq b \\ 0, & otherwise\end{cases}$$

$$=\begin{cases}\frac{1}{\frac{1}{2}-0}=2 & ; \quad 0<x<\frac{1}{2} \\ 0 & \end{cases}$$

From the definition of continuous probability distribution,

$$E(x)=\int_{-\infty}^{\infty} x\, f(x)dx=\int_{0}^{\frac{1}{2}} x\cdot 2\, dx=\frac{1}{4}$$

Gamma Distribution:

1. Putting α=1 in Gamma distribution results in _______
a) Exponential Distribution
b) Normal Distribution
c) Poisson Distribution
d) Binomial Distribution

Ans: A

Answer: a
Explanation: $f(x) = \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x} / \Gamma(\alpha)$ for $x > 0$
= 0 otherwise
If we let α=1, we obtain
$f(x) = \lambda e^{-\lambda x}$ for $x > 0$
= 0 otherwise.
Hence we obtain Exponential Distribution.

2. The lifetime of a component follows a Gamma distribution with a mean of 500 hours and a variance of 100,000 hours^2. Find the shape and rate parameters.

Ans:

$\alpha = 2.5$

$\beta = 0.005$

Continuous

PRACTICE QUESTION – EXTRAS

We have the equations: $\alpha/\beta = 500$ and $\alpha/\beta^2 = 100,000$.

Dividing the second equation by the first gives $1/\beta = 100,000/500 = 200$, so $\beta = 1/200 = 0.005$.

Substituting β back into the mean equation: $\alpha/(1/200) = 500 \implies \alpha = 500 * (1/200) = 2.5$.

3. For a Gamma distribution with α=2 and rate β=3, what is the probability that X is less than 4?

Ans: 0.9999

This calculation builds on the transformation used previously, where `2βX` follows a chi-square distribution with `2α` degrees of freedom (`df`).

- **Degrees of Freedom (df):** $df = 2 * \alpha = 4$
- **Transformation:** The probability `P(X < 4)` is transformed into the chi-square equivalent:
  - $P(2 \times 3 \times X < 2 \times 3 \times 4)$
  - $P(\chi^2(4) < 24)$
- **Calculation:** This is the value of the chi-square CDF at 24. It can also be found by taking 1 minus the tail probability.
  - $P(\chi^2(4) < 24) = 1 - P(\chi^2(4) > 24)$
  - Using the previously determined value, this is approximately $1 - 0.0000795 = 0.9999205$.

Exponential Distribution:

1. A mobile conversation follows a exponential distribution f (x) = (1/3)$e^{-x/3}$. What is the probability that the conversation takes more than 5 minutes?

Ans: $e^{-5/3}$

X follows exponential with λ =1/3

P(X > 5) = 1- P(X < 5)

=1 – (1-$e^{-5*1/3)}$

Continuous

PRACTICE QUESTION – EXTRAS

2. Let X be an exponential random variable with parameter $\lambda = \ln 3$. Compute the following probability: P(2<X < 4)

Ans: 0.9877

First of all we can write the probability as

$$\begin{aligned} P(2 \leq X \leq 4) &= P(\{X = 2\} \cup \{2 < X \leq 4\}) \\ &= P(X = 2) + P(2 < X \leq 4) \\ &= P(2 < X \leq 4) \end{aligned}$$

using the fact that the probability that a continuous random variable takes on any specific value is equal to zero (see Continuous random variables and zero-probability events). Now, the probability can be written in terms of the distribution function of X as

$$\begin{aligned} P(2 \leq X \leq 4) &= P(2 < X \leq 4) \\ &= F_X(4) - F_X(2) \\ &= [1 - \exp(-\ln(3) \cdot 4)] - [1 - \exp(-\ln(3) \cdot 2)] \\ &= \exp(-\ln(3) \cdot 2) - \exp(-\ln(3) \cdot 4) \\ &= 3^{-2} - 3^{-4} \end{aligned}$$

3. What is the probability that a random variable Xis less than its expected value, if X has an exponential distribution with parameter λ?

Ans: 0.63212

The expected value of an exponential random variable with parameter λ is

$$E[X] = \frac{1}{\lambda}$$

The probability above can be computed by using the distribution function of X:

$$\begin{aligned} P(X \leq E[X]) &= P\left(X \leq \frac{1}{\lambda}\right) \\ &= F_X\left(\frac{1}{\lambda}\right) \\ &= 1 - \exp\left(-\lambda \frac{1}{\lambda}\right) \\ &= 1 - \exp(-1) \end{aligned}$$

Normal distribution

1. The annual precipitation data of a city is normally distributed with mean and standard deviation as 1000 mm and 200 mm , respectively. The probability that the annual precipitation will be more than 1200 mm is

Ans: 0.15866

Concept:

- The normal distribution is a probability function that describes how the values of a variable are distributed.
- For a normally distributed variable x with mean μ and standard deviation σ, the normal variate z is given by the formula: $z = \frac{x-\mu}{\sigma}$.

Calculation:

Given

Standard deviation σ = 200 mm, Mean μ = 1000 mm.

For x = 1200, $z = \frac{x-\mu}{\sigma} = \frac{1200-1000}{200} = 1$

P (X > 1200 mm) = P (z > 1)

z is normal variate,

We know (P (- 1 < Z < 1) = 0.68 (i.e. 68% of data is within one standard deviation of mean)

P (0 < Z < 1) = 0.68/2 = 0.34

So P (z >1) = 0.5 - 0.34 = 0.16 % < 50 %

PZ < 1) = 0.84134

2. The weights of bags of red gravel may be modelled by a normal distribution with mean 25.8 kg and standard deviation 0.5 kg .

(a) Determine the probability that a randomly selected bag of red gravel will weigh:
(i) less than 25 kg ;

(ii) between 25.5 kg and 26.5 kg

(b) Determine, to two decimal places, the weight exceeded by 75% of bags.

Ans: a)i)0.055, ii) 0.645 b) 25.46

Continuous

PRACTICE QUESTION – EXTRAS

	Weights $W \sim N(25.8, 0.5^2)$
(a)(i)	$P(W < 25) = P\left(Z < \frac{25.0 - 25.8}{0.5}\right)$
	$= P(Z < -1.6) = P(Z > 1.6) = 1 - \Phi(1.6)$
	$1 - 0.94520 = 0.05480 \Rightarrow 0.054 \text{ to } 0.055$
(ii)	$P(25.5 < W < 26.5)$ $P\left(\frac{25.5 - 25.8}{0.5} < Z < \frac{26.5 - 25.8}{0.5}\right) =$
	$P(-0.6 < Z < 1.4) =$
	$\Phi(1.4) - (1 - \Phi(0.6)) =$ $0.91924 - 1 + 0.72575 = 0.64499 \Rightarrow$ 0.644 to 0.646
(b)	$P(W > w) = 75\%$
	$z_{0.75} = -0.6745$
	and $z = \frac{w - 25.8}{0.5}$
	$\therefore \frac{w - 25.8}{0.5} = -0.6745$
	$\therefore$ $w = 25.46$ to 25.47

Continuous

PRACTICE QUESTION – EXTRAS

3. The length, X centimetres, of eels in a river may be assumed to be normally distributed with mean 48 and standard deviation 8 .

An angler catches an eel from the river. Determine the probability that the length of the eel is:
(a) exactly 60 cm ;
(b) less than 60 cm ;
(c) within 5% of the mean length.

Q	Solution
1(a)	Length $X \sim \mathrm{N}(48, 8^2)$
	$\mathrm{P}(X = 60) = 0$
(b)	$\mathrm{P}(X < 60) = \mathrm{P}\left(Z < \frac{60-48}{8}\right)$
	$= \mathrm{P}(Z < 1.5) = 0.933$
(c)	5% of 48 = 2.4
	$\mathrm{P}(48 - x < X < 48 + x)$
	$= \mathrm{P}(-0.3 < Z < 0.3)$
	$= \Phi(0.3) - (1 - \Phi(0.3))$
	$= 2 \times 0.61791 - 1$
	$= 0.235$ to 0.236

Log-normal Distribution

1. Suppose the lifetime of a motor has a lognormal distribution. What is the probability the lifetime exceeds 12,000 hours if the mean and standard deviation of the normal random variable are 11 hours and 1.3 hours, respectively?

$\mu = 11, \quad \sigma = 1.3, \quad P(X > 12{,}000) = ?$

$$P(X \geq 12{,}000) = 1 - P(X < 12{,}000)$$

$$P(X > 12{,}000) = 1 - \Phi\left(\frac{\ln 12000 - 11}{1.3}\right)$$

$$P(X > 12{,}000) = 1 - P\left(z \leq \left(\frac{9.393 - 11}{1.3}\right)\right)$$

$$P(X > 12{,}000) = 1 - P(z < -1.236)$$

$$P(X > 12{,}000) = 1 - (0.10823)$$

$$P(X > 12{,}000) = 0.8918$$

2.

X : battery's lifespan , follows log normal distribution with parameters μ=2 and σ=0.5, find the mean and variance of the battery's lifespan.

Calculate the mean: Use the mean formula for a log-normal distribution:

$$E(X) = e^{\mu + \frac{\sigma^2}{2}} = e^{2 + \frac{0.5^2}{2}} = e^{2 + \frac{0.25}{2}} = e^{2 + 0.125} = e^{2.125} \approx 8.372$$

Calculate the variance: Use the variance formula for a log-normal distribution:

$$Var(X) = (e^{\sigma^2} - 1)e^{2\mu + \sigma^2} = (e^{0.5^2} - 1)e^{2(2) + 0.5^2} = (e^{0.25} - 1)e^{4.25}$$

$$Var(X) \approx (1.284 - 1)(70.108) = (0.284)(70.108) \approx 19.91$$

3. The sales revenue for a company, X, is log-normally distributed. It has an arithmetic mean of 149.157 and a variance of 223.5945.

Ans: μ= 5 and σ=0.10

$$Var(X) = (e^{\sigma^2} - 1)[e^{\mu+\frac{\sigma^2}{2}}]^2 = (e^{\sigma^2} - 1)[E(X)]^2.$$
$$223.5945 = (e^{\sigma^2} - 1)(149.157)^2$$
$$223.5945 = (e^{\sigma^2} - 1)(22247.96)$$
$$e^{\sigma^2} - 1 = \frac{223.5945}{22247.96} \approx 0.01005$$
$$e^{\sigma^2} \approx 1.01005$$
$$\sigma^2 \approx \ln(1.01005) \approx 0.01$$
$$\sigma = \sqrt{0.01} = 0.1$$

Solve for μ: Use the mean equation:

$$149.157 = e^{\mu+\frac{\sigma^2}{2}}$$
$$\ln(149.157) = \mu + \frac{\sigma^2}{2}$$
$$5.005 \approx \mu + \frac{0.01}{2}$$
$$5.005 \approx \mu + 0.005$$
$$\mu = 5$$

Beta Distribution

1. A random variable X follows a Beta distribution with parameters α=3 and β=5. Calculate its mean and variance.

Mean: Using the formula $E[X] = \frac{\alpha}{\alpha+\beta}$, we get:

$$E[X] = \frac{3}{3+5} = \frac{3}{8} = 0.375$$

Variance: Using the formula $\mathrm{Var}(X) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$, we get:

$$\mathrm{Var}(X) = \frac{3 \cdot 5}{(3+5)^2(3+5+1)} = \frac{15}{(8)^2(9)} = \frac{15}{64 \cdot 9} = \frac{15}{576} \approx 0.026$$

2. The proportion of successful product launches for a company is modeled by a Beta distribution. From historical data, the mean proportion of successful launches is found to be 0.6, and the variance is 0.04. Determine the parameters α and β for this Beta distribution.

Ans:

Use the mean formula to establish a relationship between α and β:

$$0.6 = \frac{\alpha}{\alpha+\beta}$$
$$0.6(\alpha+\beta) = \alpha$$
$$0.6\alpha + 0.6\beta = \alpha$$
$$0.6\beta = 0.4\alpha$$
$$\beta = \frac{0.4}{0.6}\alpha \Longrightarrow \beta = \frac{2}{3}\alpha$$

Substitute this relationship into the variance formula:

$$0.04 = \frac{\alpha(\frac{2}{3}\alpha)}{(\alpha+\frac{2}{3}\alpha)^2(\alpha+\frac{2}{3}\alpha+1)}$$
$$0.04 = \frac{\frac{2}{3}\alpha^2}{(\frac{5}{3}\alpha)^2(\frac{5}{3}\alpha+1)}$$
$$0.04 = \frac{\frac{2}{3}\alpha^2}{(\frac{25}{9}\alpha^2)(\frac{5}{3}\alpha+1)}$$

Simplify and solve for α:

$$0.04 = \frac{\frac{2}{3}\alpha^2}{(\frac{25}{9}\alpha^2)(\frac{5\alpha+3}{3})}$$

Multiply both sides by $(\frac{25}{9}\alpha^2)(\frac{5\alpha+3}{3})$:

$$0.04 \cdot \frac{25}{9}\alpha^2 \cdot \frac{5\alpha+3}{3} = \frac{2}{3}\alpha^2$$
$$0.04 \cdot \frac{25}{9}\alpha^2 \cdot (5\alpha+3) = 2\alpha^2$$

Since α must be positive, we can divide both sides by α^2:

$$0.04 \cdot \frac{25}{9} \cdot (5\alpha+3) = 2$$
$$1 \cdot (5\alpha+3) = 2 \cdot 9$$
$$5\alpha + 3 = 18$$
$$5\alpha = 15$$
$$\alpha = 3$$

Find β using the relationship from step 1:

$$\beta = \frac{2}{3}\alpha = \frac{2}{3}(3) = 2$$

Continuous

PRACTICE QUESTION – EXTRAS