Lecture 1

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

**Class:** MSc

**Subject :** Pricing & Reserving for Life Insurance Products

**Subject Code:**

**Chapter:** Unit 1 Chapter 1

**Chapter Name:** Transition Intensities and Multiple decrement model

# Today’s Agenda

1. Introduction
   1. The basic two state model
   2. The multiple state model

2. Valuing continuous cash flows using multiple state models

1. Defining the multiple state models

1. Multiple Decrement models
   1. Multiple decrement probabilities
   2. Single decrement model
   3. Deriving probabilities from transition intensities
   4. Multiple decrement tables
   5. Integral formulae for multiple decrement probabilities

2. Valuing cash flows

# 1 Introduction

**So far we have considered contingencies where a life is exposed to death only. If we suppose that a life is subject to more than one transition, then the transitions are referred to as a set of *competing risks*.**

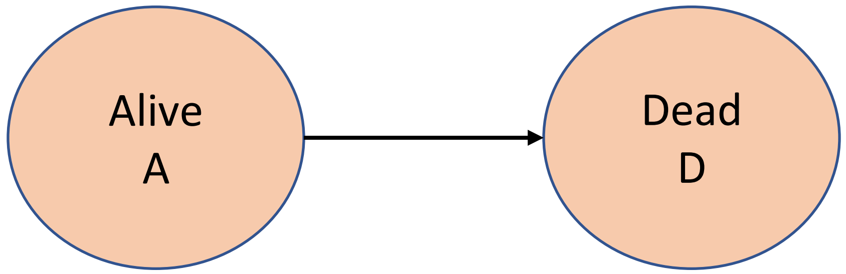
For example, a member of a pension scheme can, in order for the associated pension scheme benefits to be valued, be regarded as exposed to the competing risks of retirement and death.

In a similar way, a person with a health insurance policy who is in good health, can be considered as exposed to the competing risks of becoming sick and dying.

# 1.1 The basic two state model

- So far, we have modelled the uncertainty over the duration of an individual's future lifetime by regarding the future lifetime as a random variable, $T_x$, for an individual currently aged $x$, with a given cumulative distribution function, $F_x\,(t)$ $(= \Pr[T_x \le t])$, and survival function, $S_x(t) = 1 - F_x(t)$.

- **This is probabilistic model** in the sense that for an individual aged $x$ we have a single random variable, $T_x$, whose distribution, and hence all associated probabilities, is assumed to be known.




- We can represent this model diagrammatically as shown in Figure. Our individual is, at any time, in one of two **states**, 'Alive' and 'Dead'.

- Transition from state A to state D is allowed, as indicated by the direction of the arrow, but transitions in the opposite direction cannot occur. This is an example of a **multiple state model** with two states.

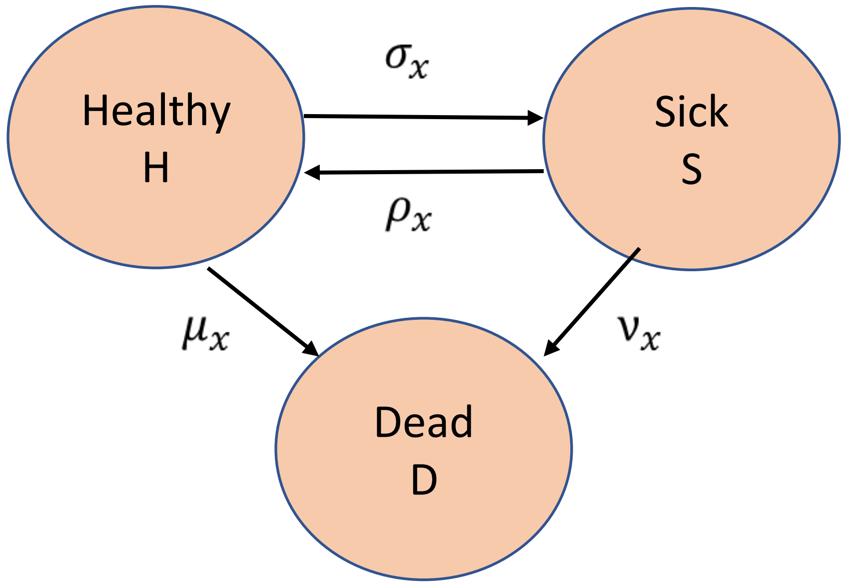
# 1.2 The multiple state model

Multiple state models are well suited to valuing cash flows that are dependent on multiple transitions, such as of health insurance contracts.

The model will be chosen to include the relevant states, and transitions between states, that are necessary to replicate the required cash flows for the contract concerned.

For example, a simple income protection policy will have **the general three-state healthy-sick-dead model suitable.**

The labels on the **arrows relate to the *transition intensities*** from one state to another. A transition intensity can also be referred to as a *transition rate* or a *force of transition*, and so is the same type of quantity as the force of mortality that we met in earlier chapters.



# Notations

Let *i* and *j* denote any two different states.

**Define $\mu_x^{ij}$ to be the transition intensity from state i to state j at age x (so, for example, $\mu_x^{HS} = \sigma_x$ in the above model).**

**Also define the related transition probability:**

**${}_tp_x^{ij}$ = P [ in state j at age x+t | in state i at age x ]**

**where now i and j need not be different.**

For example, for the model shown above:

**${}_tp_x^{HD}$ represents the probability that a life in the healthy state at age x will be in the dead state at age x + t .** This probability encompasses all possible routes from *healthy* to *dead*, which may or may not include one or more visits to the *sick* state.

# Question

What does $tp_x^{ss}$ represent??

# Notations

The event whose probability is defined by the expression:

**$_tp_x^{ij}$ = P [ in state j at age x+t | in state i at age x ]**

**does not specify what must happen between age x and age x + t , however. In particular, if i = j , it does not require that the life remains in state i between these ages.**

**So for any state i , also define the related transition probability:**

**$_tp_x^{\overline{ii}}$ = P [ in state i from age x to x+t | in state i at age x ]**

This is sometimes referred to as the occupancy probability, as it relates to the probability of staying in (or occupying) state i from age x to age x + t .

# Notations

The differential equation for $tp_x^{\overline{ii}}$ has the closed form solution:

$$\mathbf{t}\boldsymbol{p}_{\boldsymbol{x}}^{\overline{\boldsymbol{ii}}} = \mathbf{exp}\ (-\int_0^t \sum_{j \neq i} \boldsymbol{\mu}_{\boldsymbol{x}+\boldsymbol{s}}^{\boldsymbol{ij}}\ \boldsymbol{ds})$$

This result is particularly important in the construction of multiple decrement models. In the formula for $tp_x^{\overline{ii}}$, the term $\mu x + sij$ relates to the *total* force of transition out of state $i$ at age $x + s$ .

# 2 Valuing continuous cash flows using multiple state models

The EPV of a lump sum of 1 payable on death (whether directly from healthy or from having first become sick) of a healthy life currently aged *x* is:

$$\int_0^{\infty} e^{-\delta t} \left( {}_tp_x^{HH} \, \mu_{x+t} + {}_tp_x^{HS} \, \nu_{x+t} \right) dt$$

(assuming a constant force of interest $\delta$ ).
This is type 1 of above.

Think and Discuss intuitively about the built up of this formula!

# 2 Valuing continuous cash flows using multiple state models

The EPV of an annuity of 1 per annum payable continuously during sickness of a healthy life currently aged $x$ is:

$$\int_0^\infty e^{-\delta t} \; {}_t p_x^{HS} \, dt$$

(assuming a constant force of interest δ ).
This is of type 2.

Think and Discuss intuitively about the built up of this formula!

# 2 Valuing continuous cash flows using multiple state models

The EPV of a premium of 1 per annum payable continuously, but waived during periods of sickness, by a healthy life currently aged $x$ is:

$$\int_0^\infty e^{-\delta t}\ {}_tp_x^{HH}\ dt$$

(assuming a constant force of interest $\delta$ ).

Think and Discuss intuitively about the built up of this formula!

# 3 Defining the multiple state models

An important actuarial skill is to be able to choose, or design, an appropriate model for a particular purpose or application. Here, we need to be able to design or select a multiple state model for the purpose of valuing cashflows, where the cashflows depend on life and/or health dependent events.

A multiple state Markov model, of the type we have been using in this chapter, is fully defined by the following:

- the possible states that can be occupied
- the possible transitions that can be made between the states (*ie* all the arrows in the transition diagram)
- the values of the forces of transition between the states at each age.

# 4 Multiple decrement models

A multiple decrement model is a multiple state model which has:

- one active state, and
- one or more absorbing exit states.

Many practical situations involving competing risks can be modelled adequately using this simplified model structure.

# 4.1 Multiple decrement probabilities

In a multiple decrement model, we only need to define two types of probability.

- The first is $\mathbf{t(aq)_x^r}$ , which is defined as the dependent probability that an individual aged x in the active state will be removed from that state between ages x and x + t by the decrement r . (By 'dependent', we mean in the presence of all other risks of decrement in the population).
- When t = 1 this is written as $(aq)_x^r$.

- The second is $\mathbf{t(ap)_x}$ , which is defined as the dependent probability that an individual aged x in the active state will still be in the active state at age x + t .
- When t = 1 this is written as $(ap)_x$.

# Question

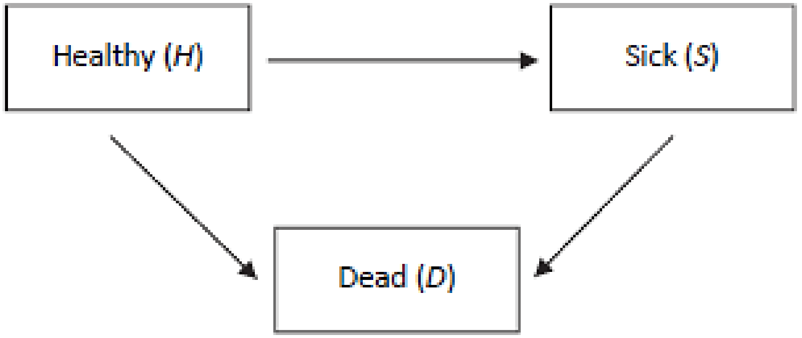
Consider the following 3-state model:

Assuming that H is the active state, explain whether or not each of the following is true, and if not, state with reasons.

a) $(aq)_x^s = {}_1p_x^{HS}$

b) $(aq)_x^d = {}_1p_x^{HD}$

c) $(aq)_x^{\cdot} = {}_1p_x^{HH}$



# Solution

a) $(aq)_x^s = 1p_x^{HS}$ -> $1p_x^{HS}$ will be smaller than $(aq)_x^s$.

b) $(aq)_x^d = 1p_x^{HD}$ -> $1p_x^{HD}$ will be larger than $(aq)_x^d$.

c) $(ap)_x^; = 1p_x^{HH}$ -> These two will be equal for this model.

## 4.2 Single decrement model

It is also useful to consider the special case of a single decrement model, which only has one cause of decrement.

For this we define **$_tq_x^r$ to be the independent probability** that an individual aged x in the active state will be removed from that state between ages x and x + t by the decrement r . (By 'independent', we mean when r is the only risk of decrement acting on the population).
When t = 1 this is written as $q_x^r$.

# 4.3 Deriving probabilities from transition intensities

**In the multiple state model, we have the following general result:**

$$\mathbf{{}_tp_x^{\overline{ii}} = exp\ (-\int_0^t \sum_{j \neq i} \mu_{x+s}^{ij}\ ds)}$$

**In the case of the multiple decrement model, in which return to the active state is not possible, we have:**

$${}_t(ap)_x = {}_tp_x^{HH} = \mathbf{{}_tp_x^{\overline{HH}}}$$

Since our double decrement model has decrements of sickness and death, we have:

$${}_t(ap)_x = \mathbf{{}_tp_x^{\overline{HH}}} = \exp\ [-\int_{s=0}^{t} \left(\boldsymbol{\mu_{x+s}^{HS}} + \boldsymbol{\mu_{x+s}^{HD}}\right) \boldsymbol{ds}\ ] = \exp\ [-\int_{s=0}^{t} (\boldsymbol{\sigma_{x+s}} + \boldsymbol{\mu_{x+s}}) \boldsymbol{ds}\ ]$$

**Therefore, assuming constant transition intensities:**

$${}_t(ap)_x = e^{-(\mu+\sigma)t}$$

# 4.3 Deriving probabilities from transition intensities

**For the other probabilities, we have the closed form solutions (of differential equation) (with *t* = 1):**

- $(aq)_x^s = \frac{\sigma}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right)$
- $(aq)_x^d = \frac{\mu}{\mu+\sigma}\left(1 - e^{-(\mu+\sigma)}\right)$

These solutions are obtained by integrating the differential equations with respect to $t$ between the limits of $t = 0$ and $t = 1$ .

Discuss the two components of the above product $(aq)_x^s$.

# 4.3 Deriving probabilities from transition intensities

Discuss the two components of the above product $(aq)_x^s$.

Note that $(aq)_x^s$ is the product of:

- $1-e^{-(\mu+\sigma)}=1-(ap)_x$, *ie* the probability that a life who is in the active state at the start of the year, is *not* in the active state at the end of the year, and
- $\dfrac{\sigma}{\mu+\sigma}$, *ie* the proportion of the total force acting on the life in the active state that relates to transitions to the sick state. This represents the conditional probability that the transition from the active state takes the life into the sick state, given that there is a transition out of the active state.

# Question

A life insurance company issues a three-year policy to a life that offers the following benefits:

- On death during the term of the policy, a sum of 37,500.
- On redundancy during the term of the policy, a return of 105% of total premiums paid.
- On surrender during the term of the policy, a return of 33% of total premiums paid.
- On survival to the end of the term, a sum of 39,000.

Premiums of 12,500 are payable annually in advance throughout the term of the policy or until earlier claim.

The death, redundancy and surrender benefits are payable immediately on claim. The policy ceases on payment of any claim.

The company uses the following basis to profit test this policy:
Independent force of mortality 1.5%
Independent force of redundancy 2%
Independent force of surrender 5% in years 1and 2 only
Interest earned on cash flows 2.5% per annum
Expenses 2.5% of each premium paid
Reserves Ignore
The company assumes that each force of decrement is constant over each year of age.
(i) Calculate the dependent rates of mortality, redundancy and surrender for each policy year.

# Solution

(i) The dependent rates of decrement are calculated for each policy year using:-

$$(aq)_x^j = \frac{\mu^j}{\mu^d + \mu^r + \mu^s}\left[1 - e^{-(\mu^d+\mu^r+\mu^s)}\right]$$

where $d$ denotes mortality, $r$ retirement and $s$ surrender

$\Rightarrow$ for policy years 1 and 2

$$(aq)^d = \frac{\mu^d}{\mu^d + \mu^r + \mu^s}\left[1 - e^{-(\mu^d+\mu^r+\mu^s)}\right] = \frac{0.015}{0.085}\left[1 - e^{-(0.085)}\right] = 0.01438$$

$$(aq)^r = \frac{\mu^r}{\mu^d + \mu^r + \mu^s}\left[1 - e^{-(\mu^d+\mu^r+\mu^s)}\right] = \frac{0.02}{0.085}\left[1 - e^{-(0.085)}\right] = 0.019174$$

$$(aq)^s = \frac{\mu^s}{\mu^d + \mu^r + \mu^s}\left[1 - e^{-(\mu^d+\mu^r+\mu^s)}\right] = \frac{0.05}{0.085}\left[1 - e^{-(0.085)}\right] = 0.047934$$

# Solution

$\Rightarrow$ for policy year 3

$$(aq)^d = \frac{\mu^d}{\mu^d + \mu^r}\left[1 - e^{-(\mu^d+\mu^r)}\right] = \frac{0.015}{0.035}\left[1 - e^{-(0.035)}\right] = 0.014741$$

$$(aq)^r = \frac{\mu^r}{\mu^d + \mu^r}\left[1 - e^{-(\mu^d+\mu^r)}\right] = \frac{0.02}{0.035}\left[1 - e^{-(0.035)}\right] = 0.019654$$

which gives the following multiple decrement table:

| Year $t$ | $\mu^d$ | $\mu^s$ | $\mu^r$ | $(aq)^d$ | $(aq)^s$ | $(aq)^r$ | $(ap)$ | ${}_{t-1}(ap)$ |
|---|---|---|---|---|---|---|---|---|
| 1 | .015 | .05 | .02 | .014380 | .047934 | .019174 | .918512 | 1.0 |
| 2 | .015 | .05 | .02 | .014380 | .047934 | .019174 | .918512 | .918512 |
| 3 | .015 | 0 | .02 | .014741 | 0 | .019654 | .956505 | .843665 |

# 4.4 Multiple Decrement Tables

**A multiple decrement table is a computational tool for dealing with a population subject to multiple decrements.**

We introduce the following notation as an extension of the (single decrement) life table approach:
$(al)x$ = active population at age $x$
and α , β , ... the labels for the types of independent decrements to which the population is subject.

Then the multiple decrement table is a numerical representation of the development of the population, such that:
**$(al)x{+}1$ = $(al)x$ - number of lives removed between ages x and x+1 due to decrement α**
**- number of lives removed between ages x and x+1 due to decrement β ........**

All the life table formulas derived earlier will hold, just that we need to take care of the multiple decrements.

**It is also conventional to write the transition intensity (or *force of decrement*) due to cause *k* at age *x* in the multiple decrement model as $(a\mu)_x^k$.**

# 4.5 Integral formula for multiple decrement probabilities

We can also obtain expressions for multiple decrement probabilities without making the assumption that forces of decrement are constant over each year of age.

For example, with a time-varying sickness intensity, the differential equation for ${}_t(aq)_x^s$ would become:

$$\frac{d}{dt}\,{}_t(aq)_x^s = {}_t(ap)_x\,\sigma_{x+t} = {}_t(ap)_x\,(a\mu)_{x+t}^s$$

Integrating over $t$ = 0 to $t$ = 1 we obtain:

$${}_1(aq)_x^s - {}_0(aq)_x^s = \int_0^1 {}_t(ap)_x\,(a\mu)_{x+t}^s\,dt$$

*As* ${}_0(aq)_x^s = 0$:

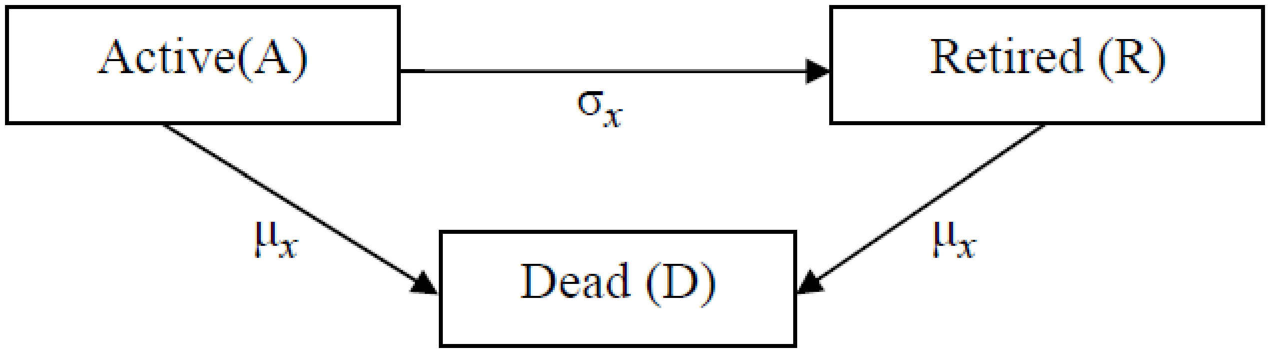
$$\boldsymbol{{}_1(aq)_x^s = \int_0^1 {}_t(ap)_x\,(a\mu)_{x+t}^s\,dt}$$

In general,

$$\boldsymbol{{}_t(aq)_x^j = \int_0^1 {}_t(ap)_x\,(a\mu)_{x+t}^j\,dt}$$

# Question

A life insurance company models the experience of its pension scheme contracts using the following three-state model:




(i) Derive the dependent probability of a life currently Active and aged $x$ retiring in the year of age $x$ to $(x + 1)$ in terms of the transition intensities. [2]

(ii) Derive a formula for the independent probability of a life currently Active and aged $x$ retiring in the year of age $x$ to $(x + 1)$ using the dependent probabilities. [4]

[Total 6]

# Solution

(i) We are looking to derive $(aq)_x^r$ in terms of $\sigma_x$ and $\mu_x$

Use the Kolmogorov equations (assuming the transition intensities are constant across a year age):

$$\frac{\partial}{\partial t}\,{}_t(aq)_x^r = \sigma e^{-(\sigma+\mu)t}$$

$$(aq)_x^r = \frac{\sigma}{(\sigma+\mu)}(1-e^{-(\sigma+\mu)})$$

# Solution

(ii) Similarly

$$(aq)_x^d = \frac{\mu}{(\sigma+\mu)}(1-e^{-(\sigma+\mu)})$$

Note that:

$$1-((aq)_x^r+(aq)_x^d) = e^{-(\sigma+\mu)}$$

$$\Rightarrow \sigma+\mu = -\log(1-((aq)_x^r+(aq)_x^d))$$

So

$$(aq)_x^r = \frac{\sigma}{(-\log(1-((aq)_x^r+(aq)_x^d)))}((aq)_x^r+(aq)_x^d)$$

this can be rearranged to show

$$-\sigma = \frac{(aq)_x^r}{(aq)_x^r+(aq)_x^d}\log(1-((aq)_x^r+(aq)_x^d))$$

Given that:

$$q_x^r = 1-e^{-\sigma},$$

then

$$q_x^r = 1-\left[1-((aq)_x^r+(aq)_x^d)\right]^{(aq)_x^r / ((aq)_x^r+(aq)_x^d)}$$

# 5 Valuing Cash flows

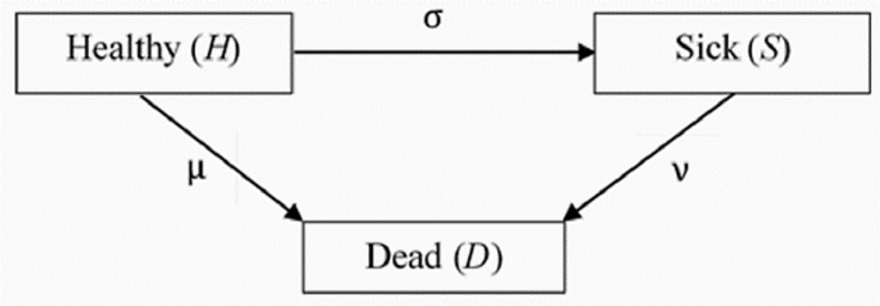
Cash flows can be evaluated either using integrals (continuously contingent on multiple decrements), or using a discrete, annual summation approach:

**EPV = $\sum$ {cashflow} x {discount factor} x {probability}**

# Question

**CM1A April 2021 Q2**

A life insurance company uses the following three-state model, with constant forces of transition, to price its stand-alone critical illness policies.




Under these policies, a lump sum benefit is payable when a life becomes critically ill during the policy term. No other benefits are payable.

A 30-year policy with sum assured $150,000 is issued to a healthy life aged 35 exact. The expected present value of the benefit at outset is given by the following formula:

$$m \times \int_a^b n \times e^{zt} \, dt.$$

# Question

(i) State the numerical values of ***a***, ***b, m***, ***n*** and ***z***. [3]

(ii) Calculate the expected present value of the benefit for this policy based on your answer to part (i).

Basis: $\mu = 0.01$
$\sigma = 0.02$
$\nu = 0.04$

Interest: 3% p.a. effective
[3]

[Total 6]

## Solution

(i)

a=0,
b=30
m= 150,000
n=0.02

$$z = -ln(1.03) - 0.01 - 0.02 = -0.059559$$

(ii)

$$= -\frac{150000 \times 0.02}{0.059559}\left[e^{-0.059559t}\right]_0^{30}$$

= 50,370.22 [1 – 0.167500]

= 41, 933.21 (without rounding 41,933.28)

# 2 Valuing continuous cash flows using multiple state models

Consider, for example, a healthy life who is subject to the competing risks of sickness and death. A multiple state model can be used to construct integral expressions for the EPVs of the following types of cash flows:

- a lump sum paid immediately on transition from one state to another (Type 1)
- an income payable while occupying a particular state (Type 2).