Lecture 2

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Pricing and Reserving for Life Insurance products – 2

Subject Code:

Chapter: Unit 2

Chapter Name: Gross premium and reserving – 2 [Variable benefits and with profit contracts]

Today's Agenda

1. Gross Premiums and Reserving – 1 (Revision)

2. Introduction
    1. Variable Payments
    2. Payments varying at a constant compound rate
    3. Payments varying at a constant monetary amount

3. Conventional with-profit contracts

1 Gross Premium & Reserving – 1 (Revision)

The gross premium is the premium required to meet all the costs under an insurance contract, and is the premium that the policyholder pays. When we talk of 'the premium' for a contract, we mean the gross premium. It is also sometimes referred to as the office Premium.

When the outgo includes benefits and expenses, and the income is the gross premiums, then L is referred to as the gross future loss random variable.

L = PV of benefit outgo + PV of expenses – PV of gross premium income

1 Gross Premium & Reserving – 1 (Revision)

Suppose we can allocate expenses as:

- *I* initial expenses in excess of those occurring regularly each year
- *e* level annual expenses
- *f* additional expenses incurred when the contract terminates

Let G denote the amount of Gross Premium**.**

The gross future loss random variable when a policy is issued to a life aged *x* is:

$$\mathbf{L = S}v^{T_x} + \mathbf{I} + \mathbf{e}\bar{a}_{\overline{T_x|}} + \mathbf{f}v^{T_x} - \mathbf{G}\bar{a}_{\overline{T_x|}}$$

where a gross premium of *G* secures a sum assured of *S* , the sum assured is paid immediately on death and the premium is payable continuously.

under the equivalence premium principle,

EPV of benefits + **EPV of expenses** = **EPV of gross premium income**.

1 Gross Premium & Reserving – 1 (Revision)

Premiums (and reserves) can be calculated which satisfy probabilities involving the gross future loss random variable.

Example

A whole life assurance pays a sum assured of 10,000 at the end of the year of death of a life aged 50 exact at entry. Assuming 3% per annum interest, AM92 Ultimate mortality and expenses of 4% of every premium, calculate the smallest level annual premium payable at the start of each year that will ensure the probability of making a loss under this contract is not greater than 5%.

If the annual premium is *G* , the future loss (random variable) of the policy at outset is:

$L_0 = 10000\ v^{K_{50}+1} - 0.96\ G\ \ddot{a}_{\overline{K_{50}+1|}}$

We need to find the smallest value of *G* such that:

$Pr(L > 0) \leq 0.05$

ie such that $Pr(L \leq 0) \geq 0.95$

1 Gross Premium & Reserving – 1 (Revision)

Define G_n to be the annual premium that ensures $L_0 = 0$ for $K_{50} = n$.

This means:

$$G_n = \frac{10000\ v^{n+1}}{0.96\ \ddot{a}_{\overline{n+1|}}}$$

Now:

P ($L_0 \le 0$| G = G_n) = P ($K_{50} \ge$ n)

However, this probability needs to be at least 0.95…

We therefore find the largest value of n that satisfies this condition, and the corresponding value of Gn is then the minimum premium required

So: **P ($K_{50} \ge$ n) ≥ 0.95**

This can be then solved with the help of tables. Then we can calculate the Premium.

A premium calculated in this way is sometimes called a percentile premium.

1 Gross Premium & Reserving – 1 (Revision)

- Under the equivalence premium principle,

EPV of benefits + **EPV of expenses** = **EPV of gross premium income**.

- **Gross Premium Prospective Reserves**

The expected present value of the future outgo (benefits and expenses)
less
the expected present value of the future income (gross premiums)

This is the **prospective reserve because it looks forward to the future cash flows of the contract**.

- **Gross Premium Retrospective Reserves**

The retrospective reserve for a life insurance contract that is in force is defined to be, for a given basis:

The accumulated value allowing for interest and survivorship of the premiums received to date
less
the accumulated value allowing for interest and survivorship of the benefits and expenses paid to date

1 Gross Premium & Reserving – 1 (Revision)

- We define the death strain in the policy year t to $t+1$ to be the random variable, DS, say,

$$\mathbf{DS} = \begin{cases} \mathbf{0} & \textbf{If the life survives to } \mathbf{t+1} \\ (\mathbf{S} - {}_{\mathbf{t+1}}\mathbf{V}) & \textbf{If the life dies in the year } [\mathbf{t}, \mathbf{t+1}) \end{cases}$$

The maximum death strain, $(\boldsymbol{S} - {}_{\mathbf{t+1}}\mathbf{V})$ **is called the death strain at risk or DSAR.**

- The EDS is the amount the company expects to pay out, in addition to the year-end reserve for a policy.
  EDS $= \boldsymbol{q_{x+t}}(\textbf{No. of policies in force})\ (\boldsymbol{S} - {}_{\boldsymbol{t+1}}\boldsymbol{V})$

- The ADS is the amount it actually pays out, in addition to the year-end reserve.

$$\boldsymbol{ADS} = \begin{cases} \mathbf{0} & \textbf{if the life survives to } \boldsymbol{t+1} \\ (\textbf{Actual number of deaths})\ (\boldsymbol{S} - {}_{\boldsymbol{t+1}}\boldsymbol{V}) & \textbf{if the life dies in the year } [\boldsymbol{t}, \boldsymbol{t+1}) \end{cases}$$

- The mortality profit is defined as:
  Mortality profit $=$ **Expected Death Strain** $-$ **Actual Death Strain**

2 Introduction

In this chapter we look at policies for which the benefit amount varies over time, including conventional with-profits contracts. In particular, we examine how to calculate the expected present value of the benefits when they increase by a constant amount or constant percentage each year.

2.1 Variable Payments

We consider now the possibility that the payment amount varies.
In the case of a benefit payable on death, let the payment be Y_x if death occurs in the year of age (x, x+1).

Assume first that the benefit is payable at the end of the year of death. The EPV of this death benefit, to a life currently aged *x* , will be given by, using life table notation:

$$Y_x \, v \, \frac{d_x}{l_x} + Y_{x+1} \, v^2 \, \frac{d_{x+1}}{l_x} + \ldots\ldots + Y_{x+t} \, v^{x+t} \, \frac{d_{x+t}}{l_x} + \ldots\ldots\ldots\ldots$$

This can also be written using summations as:

$$\sum_{j=0}^{\infty} Y_{x+j} \, v^{x+j} \, \frac{d_{x+j}}{l_x}$$

Equivalent integral expressions apply if the benefit is payable immediately on death.

$$\sum_{j=0}^{\infty} Y_{x+j} \int_{j}^{j+1} v^t \, tp_x \, \mu_{x+t} \, dt$$

2.1 Variable Payments

Corresponding to the assurance evaluation above, we will discuss a similar approach to evaluating annuity benefits where the variation follows one of the patterns just described.

In general, the EPV of an annuity of amount F_{x+t} payable on survival to age $x + t$ to a life currently aged x , assuming, for example, immediate annual payments in arrears, would be evaluated directly from:

$$\boldsymbol{F_{x+t}\ \mathrm{v}\ \frac{l_{x+1}}{l_x} + F_{x+2}\ v^2\ \frac{l_{x+2}}{l_x} + \ldots\ldots + F_{x+t}\ v^t\ \frac{l_{x+t}}{l_x} + \ldots..}$$

Having evaluated the appropriate assurance and annuity factors, the equivalence principle may then be used to calculate the required premiums and reserves.

2.2 Payments varying at a constant compound rate

Consider first a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $(1+b)^k$ if death occurs in the year of age $(x+k, x+k+1)$, $k = 0,1,...$

The (random) present value of these benefits is:
$(\mathbf{1}+\boldsymbol{b})^{\boldsymbol{K_x}}\ \boldsymbol{v}^{\boldsymbol{K_x}+\mathbf{1}}$

Then the EPV of these benefits is:
$\sum_{k=0}^{\infty}(\mathbf{1}+\boldsymbol{b})^{\boldsymbol{k}}\ \boldsymbol{v}^{\boldsymbol{k}+\mathbf{1}}\ \boldsymbol{k}|\boldsymbol{q_x} = \frac{\mathbf{1}}{\mathbf{1}+\boldsymbol{b}}\ \boldsymbol{A_x^j}$

where the assurance function is determined on the normal mortality basis but using an interest rate, j , where:
J = $\frac{(1+i)}{(1+b)} - 1$

A similar approach may be derived for other types of assurance.

Where b is negative this approach may be used to allow for compound-decreasing benefits.

2.2 Payments varying at a constant compound rate

To consider the evaluation of compound-varying survival benefits, consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x + k$ being $(1+c)^k$, k = 1,2,... .

The (random) present value of these benefits is:

$$\sum_{k=1}^{K_x}(1+c)^k\ v^k$$

Then the EPV of this annuity is:

$$\sum_{k=1}^{\infty}(1+c)^k\ v^k\ kp_x = a_x^j$$

where the annuity function is determined on the normal mortality basis but using an interest rate j , where

$$J = \frac{(1+i)}{(1+c)} - 1$$

Where *c* is negative this approach may be used to allow for compound-decreasing annuities.

Question

On 1 January 2000, a life insurance company issued 25-year increasing term assurance policies to single lives aged 40 exact.

The death benefit, payable at the end of the year of death, was 50,000 in the first policy year and increased at the beginning of each policy year at a rate of 1.92308% per annum compound. The first increase was at the start of the second policy year.

A return of premiums paid, with no interest, is payable on survival to the end of the term of the policy.

Level premiums on the policies are payable annually in advance for 25 years or until earlier death.

The company calculates its reserves on a net premium basis and negative reserves are permitted.

(i) Show that the annual net premium for each policy is approximately equal to 323 using the basis below.

Basis:
Mortality AM92 Select
Interest 6% per annum
Expenses Ignore

Solution

(i)

Let P be the annual net premium for the increasing term assurance policy. Then the equation of value is given by:

$$P = \frac{\frac{50,000}{1.0192308} A^{1}_{[40]:\overline{25|}} @ 4\% + 25PA_{[40]:\overline{25|}}^{1} @ 6\%}{\ddot{a}^{@6\%}_{[40]:\overline{25|}}} \qquad [2½]$$

where at 4%

$$A^{1}_{[40]:\overline{25|}} = A_{[40]:\overline{25|}} - v^{25}\ {}_{25}p_{[40]}$$

$$= 0.38896 - 0.37512 \times \frac{8821.2612}{9854.3036} = 0.38896 - 0.33580 = 0.05316$$

Solution

and at 6%

$$A_{[40]:\overset{1}{\overline{25}|}} = v^{25}\ {}_{25}p_{[40]}$$

$$= 0.233 \times \frac{8821.2612}{9854.3036} = 0.20857$$

[½]

$$\Rightarrow P = \frac{\frac{50,000}{1.0192308} \times 0.05316 + 25P \times 0.20857}{13.29} \Rightarrow P = \frac{2607.849}{8.07575} = 322.95$$ [½]

2.3 Payments varying at a constant monetary amount

Whole Life Assurance

Consider, for example, a whole life assurance issued to a life aged x where the benefit, payable at the end of the year of death, is $k + 1$ if death occurs in the year of age $(x + k, x + k + 1)$, $k = 0,1,\ldots$.

The random present value is:

$$(K_x + 1)v^{K_x+1}$$

The EPV of this assurance benefit is then:

$$\sum_{k=0}^{\infty}(k + 1)v^{(k+1)}\ {}_{k|}q_x$$

which is given the actuarial symbol $(IA)_x$

Do you think we would have decreasing whole life assurance??

2.3 Payments varying at a constant monetary amount

Term Assurance

An increasing temporary assurance, with term *n* years can now be evaluated using the formula:

$$(IA)^{1}_{x:\overline{n}|} = (IA)_x - v^n \frac{l_{x+n}}{l_x}\left[(IA)_{x+n} + n\,A_{x+n}\right]$$

Discuss how this formula works??

2.3 Payments varying at a constant monetary amount

Endowment Assurance

An increasing endowment assurance, with term *n* years can be defined as:

$$(IA)_{x:\overline{n|}} = (IA)^{1}_{x:\overline{n|}} + n\,A_{x:\overset{1}{\overline{n|}}}$$

2.3 Payments varying at a constant monetary amount

Increasing assurances payable immediately on death

Increasing assurances payable immediately on death can also be defined. For example, $(IA)_x$ is the expected present value of a payment of $k + 1$ paid immediately on death occurring in the year of age $(x + k, x + k + 1), k = 0,1,...$

It can be calculated using the usual approximations, for example:

$$(I\bar{A})_x \approx (1 + i)^{1/2}\ (IA)_x$$

2.3 Payments varying at a constant monetary amount

Annities

- **Whole life annuity payable annually in arrears**

In the case of an annuity that increases by a constant amount each year consider, for example, an immediate annuity payable annually in arrears, with the benefit payable on survival to age $x + k$ being k , $k = 1,2,\ldots$.
The EPV of this annuity benefit is: $\sum_{k=1}^{\infty} k\, v^k \ {}_kp_x = (Ia)_x$

- **Temporary annuities**

Increasing temporary annuities can now be evaluated. For example, an increasing temporary annuity-due has an EPV given by:

$(I\ddot{a})_{x:\overline{n|}} = (I\ddot{a})_x - v^n \frac{l_{x+n}}{l_x} [(I\ddot{a})_{x+n} + n\, \ddot{a}_{x+n}]$

- **Annuities payable continuously**

EPV of an immediate annuity payable continuously, with the (level) benefit payable over the year of age $(x + k, x + k + 1)$ being $1 + k$, $k = 0, 1, \ldots$. The approximate calculation of this is:

$(I\bar{a})_x \approx (I\ddot{a})_x - 1/2\ddot{a}_x$

Question

A life insurance company offers an increasing term assurance that provides a benefit payable at the end of the year of death of 10,000 in the first year, increasing by 100 on each policy anniversary.

Calculate the single premium for a five year policy issued to a life aged 50 exact.

Basis:
Rate of interest 4% per annum
Mortality AM92 Select
Expenses Nil
[4]

Solution

$$EPV = (10,000-100)A^{1}_{[50]:\overline{5}|} + 100(IA)^{1}_{[50]:\overline{5}|}$$

$$= 9,900(A_{[50]} - v^5\,{}_5p_{[50]}A_{55}) + 100((IA)_{[50]} - v^5\,{}_5p_{[50]}(5A_{55} + (IA)_{55}))$$

$$= 9,900\left(0.32868 - v^5\frac{9557.8179}{9706.0977}*0.38950\right)$$

$$+100*\left(8.5639 - v^5\frac{9557.8179}{9706.0977}(5*0.38950 + 8.57976)\right)$$

$$= 132.96 + 4.34$$

$$= 137.30$$

3 Conventional with-profit contracts

In this section we see how conventional with-profits business operates.

- A conventional whole life or endowment policy can be issued on a without-profit or a with-profits basis. On a without-profit basis, both the premiums and benefits under the policy are usually fixed and guaranteed at the date of issue.

- On a with-profits basis the premiums and/or the benefits can be varied to give an additional benefit to the policyholder in respect of any emerging surplus of assets over liabilities following a valuation. For example, surplus might be used to reduce the premium payable for the same benefit or to increase the sum assured without any additional premium becoming payable.

- An alternative way of distributing surplus is to make a cash payment to policyholders.

- Where surplus is distributed so as to increase benefits, additions to the sum assured are called bonuses.

3

Conventional with-profit contracts

- Why do you think the company would declare bonuses?
- Wouldn’t it make the payments more uncertain?
- Is bonus beneficial for the company?? How does company afford bonuses?

3.1 Types of bonus

Various methods of allocating bonuses have been developed, each intended to provide a way of matching the surplus emerging over the duration of the policy.

Once added to the sum assured, bonuses become guaranteed benefits, which then need to be reserved for.

- Reversionary bonuses.
  - Bonuses are usually allocated annually, which is likely to tie in with the minimum required frequency of valuation for each insurer.
- Terminal bonuses
  - Bonuses are allocated when a policy matures or becomes a claim as a result of the death of the life assured. Terminal bonuses are usually allocated as a percentage of the basic sum assured and the bonuses allocated prior to termination

3.1 Types of bonus

Typically, bonuses are added by a mixture of annual and terminal components. The annual bonuses will be at variable rates determined from time to time by the insurer based on actual arising surpluses. These bonuses are typically added according to one of the following methods:

- Simple
  - The rate of bonus each year is a percentage of the initial (basic) sum assured under the policy. The sum assured will increase linearly over the term of the policy.
- Compound
  - The rate of bonus each year is a percentage of the basic sum assured and the bonuses added in the past. The sum assured increases exponentially over the term of the policy.
- Super-Compound
  - Two compound bonus rates are declared each year. The first rate (usually the lower) is applied to the basic sum assured. The second rate is applied to the bonuses added to the policy in the past. The sum assured increases exponentially over the term of the policy.

3.1 Types of bonus

Annual bonuses are an allocation in arrears to reflect the growth in the available surplus since the last valuation. Where annual bonuses are given, it is usual at each valuation to declare a rate of **interim bonus** which will be applied to policies becoming claims before the next valuation. This provides an allocation of bonus for the period from the last valuation to the date of the claim. This rate is applied in the same way as that used for annual bonuses.

Question

A life insurance company issues a 30-year with-profits endowment assurance policy to a life aged 35 exact. The sum assured is 100,000 together with any attaching bonuses and is payable immediately on death.
Level premiums are payable monthly in advance, ceasing on maturity or on the policyholder's death if earlier.
Simple annual bonuses are added at the end of each policy year. The death benefit does not include any bonus relating to the policy year of death.

The company calculates the premium on the following basis:
Mortality AM92 Select
Rate of interest 4% per annum
Expenses – Initial 325
– Renewal 70 at the start of the second and subsequent policy years and payable until death
– Claim 275 on death
Commission – Initial 70% of the total premium payable in the first policy year
– Renewal 2.5% of the second and subsequent monthly premiums
Bonuses Simple bonus of 2.5% of basic sum assured per annum

(i) Show that the monthly premium for this policy is approximately 292. [9]

Solution

(i) If the monthly premium and sum assured are denoted by P and S respectively then:

$$0.975 \times 12P\ddot{a}^{(12)}_{[35]:\overline{30|}} + 0.025P$$

$$= (0.975S + 275)\bar{A}^{\,1}_{[35]:\overline{30|}} + Sv^{30}\,{}_{30}p_{[35]} + 0.025S(I\bar{A})_{[35]:\overline{30|}}$$

$$+325 + 70(\ddot{a}_{[35]:\overline{30|}} - 1) + 0.7 \times 12P \qquad [6]$$

where $(I\bar{A})_{[35]:\overline{30|}} = (I\bar{A})^{\,1}_{[35]:\overline{30|}} + 30v^{30}\,{}_{30}p_{[35]}$ [½]

$$\Rightarrow 0.975 \times 12P\ddot{a}^{(12)}_{[35]:\overline{30|}} + 0.025P$$

$$= (1.04)^{0.5}\left[(0.975 \times 100,000 + 275)A^{\,1}_{[35]:\overline{30|}} + 0.025 \times 100,000(IA)^{\,1}_{[35]:\overline{30|}}\right]$$

$$+(1 + 30 \times 0.025) \times 100,000v^{30}\,{}_{30}p_{[35]} + 325 + 70(\ddot{a}_{[35]:\overline{30|}} - 1) + 8.4P$$

Solution

where

$$\ddot{a}^{(12)}_{[35]:\overline{30}|} = \ddot{a}^{(12)}_{[35]} - v^{30}\,{}_{30}p_{[35]}\ddot{a}^{(12)}_{65}$$

$$= \left(\ddot{a}_{[35]} - \frac{11}{24}\right) - v^{30}\,{}_{30}p_{[35]}\left(\ddot{a}_{65} - \frac{11}{24}\right)$$

$$= \left(21.006 - \frac{11}{24}\right) - .30832 \times \frac{8821.2612}{9892.9151}\left(12.276 - \frac{11}{24}\right)$$

$$= 20.548 - 3.249 = 17.299$$ [½]

Solution

$$A^{\;\;1}_{[35]:\overline{30|}} = A_{[35]:\overline{30|}} - v^{30}\,{}_{30}p_{[35]} == 0.32187 - 0.27492 = 0.04695 \qquad [½]$$

$$(IA)^{\;\;1}_{[35]:\overline{30|}} = (IA)_{[35]} - v^{30}\,{}_{30}p_{[35]}\left(30A_{65} + (IA)_{65}\right)$$

$$= 7.47005 - 0.30832 \times \frac{8821.2612}{9892.9151}(30 \times 0.52786 + 7.89442)$$

$$= 7.47005 - 6.52394 = 0.94611$$

[1]

$$\Rightarrow (0.975 \times 12 \times 17.299 + 0.025)P$$

$$= (1.04)^{0.5}\left[97,775 \times 0.04695 + 2,500 \times 0.94611\right]$$

$$+175,000 \times 0.27492 + 325 + 70 \times 16.631 + 8.4P$$

Solution

$$\Rightarrow 202.423P = (1.04)^{0.5}[4,590.536 + 2,365.275]$$

$$+48,111.0 + 325 + 1,164.17 + 8.4P$$

$$\Rightarrow 194.023P = 7,093.563 + 48,111.0 + 325 + 1,164.17 \Rightarrow P = 292.20 \qquad [½]$$

Question

Question continued

As at the end of the 28th policy year, the total actual past bonus additions to the policy have followed the assumptions stated in the premium basis above.

(ii) Calculate the gross prospective policy value at the end of the 28th policy year.

Policy value basis:
Mortality AM92 Ultimate
Rate of interest 4% per annum
Expenses
Renewal 85 at the start of each policy year and payable until death
Claim 300 on death
Commission
Renewal 2.5% of the monthly premiums
Bonuses Simple bonus of 2.75% of basic sum assured per annum [6]

Solution

(ii) Gross prospective policy value (calculated at 4%) is given by:

$$V^{\text{prospective}} = 170{,}000\bar{A}_{63:\overline{2|}} + (0+300)q_{63}v^{0.5} + (2750+300)\,p_{63}q_{64}v^{1.5}$$

$$+5500\,{}_2p_{63}v^2 + 85\ddot{a}_{63:\overline{2|}} - 0.975\times 12P\ddot{a}^{(12)}_{63:\overline{2|}} \qquad [4]$$

where $B = 28\times 0.025\times 100{,}000 = 70{,}000$ [½]

$$\ddot{a}^{(12)}_{63:\overline{2|}} = \ddot{a}^{(12)}_{63} - v^2\,{}_2p_{63}\ddot{a}^{(12)}_{65}$$

$$= \left(\ddot{a}_{63} - \frac{11}{24}\right) - v^2\,{}_2p_{63}\left(\ddot{a}_{65} - \frac{11}{24}\right)$$

$$= \left(13.029 - \frac{11}{24}\right) - .92456\times\frac{8821.2612}{9037.3973}\left(12.276 - \frac{11}{24}\right)$$

$$= 12.57067 - 0.90245\times 11.81767 = 1.90582$$

Solution

$$\bar{A}_{63:\overline{2|}} = (1.04)^{0.5} A^{1}_{63:\overline{2|}} + v^2 \, {}_2p_{63} = (1.04)^{0.5}(0.92498 - 0.90245) + 0.90245$$

$$= 0.92543$$

[½]

$$V^{\text{prospective}} = 170,000 \times 0.92543 + 300 \times 0.011344 \times 0.98058$$

$$+3050 \times 0.98866 \times 0.012716 \times 0.94287$$

$$+5500 \times 0.90245 + 85 \times 1.951 - 0.975 \times 12 \times 292.20 \times 1.90582$$

$$= 157323.1 + 3.3371 + 36.1534 + 4963.475 + 165.835 - 6515.50$$

$$= 155,976.39$$

[½]

[Total 15]