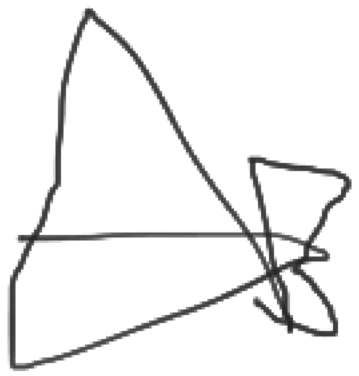
n Sample N ~ mean μ var σ^2

(i) $\frac{\Sigma(X_i - \bar{X})^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$

$S^2 = \frac{1}{n-1} \Sigma(x_i - \bar{x})^2$

(ii) $\hat{\sigma}^2 = \frac{1}{n+b} \Sigma(X_i - \bar{X})^2$

bias $\hat{\sigma}^2$ (unbiased $b = -1$)



$\frac{(n-1)S^2}{\sigma^2} = \frac{\Sigma(x_i - \bar{x})^2}{\sigma^2}$

$$\text{bias}(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2$$

$$E\left[\frac{\Sigma(x_i - \bar{x})^2}{\sigma^2}\right] = n-1$$

From $\Sigma(x_i - \bar{x})^2 = (n+b)\hat{\sigma}^2$

$$E\left(\frac{(n+b)\hat{\sigma}^2}{\sigma^2}\right) = n-1 \Rightarrow \frac{(n+b)}{\sigma^2} E(\hat{\sigma}^2) = (n-1)$$

$$E(\hat{\sigma}^2) = \frac{(n-1)}{n+b}\sigma^2$$

$$\text{bias}(\hat{\sigma}^2) = E(\hat{\sigma}^2) - \sigma^2 = \left(\frac{n-1}{n+b}\right)\sigma^2 - \sigma^2$$

$$= \frac{\cancel{n\sigma^2} - \sigma^2 - \cancel{n\sigma^2} - b\sigma^2}{n+b} = -\frac{(1+b)\sigma^2}{n+b}.$$

$$\text{if } b = -1 \Rightarrow -\frac{(1-1)\sigma^2}{n-1} = 0$$

$$MSE(\hat{\sigma}^2) = \frac{2(n-1) + (1+b)^2 \cdot \sigma^4}{(n+b)^2}$$

$$Var\left(\frac{\sum(x_i - \bar{x})^2}{\sigma^2}\right) = 2(n-1) \quad \text{—} \quad \chi^2 \; Var$$

$$Var\left(\frac{(n+b)\hat{\sigma}^2}{\sigma^2}\right) = 2(n-1) \Rightarrow \frac{(n+b)^2}{\sigma^4} Var(\hat{\sigma}^2) = 2(n-1)$$

$$\mathrm{Var}(\hat{\sigma}^2) = \frac{2(n-1)\sigma^4}{(n+b)^2}$$

$$\text{bias}^2 = \frac{(1+b)^2\sigma^4}{(n+b)^2}$$

+

MSE =

MSE = ? 0

$n \to \infty$

consistent

PQ4 - Unit 1 Apr 2008 Q 11

(i) no. of samples n, k policies expire & exp is stopped $0 < k < n$

$\exp(\lambda)$ = dist^n

$P(T > \underline{\underline{t_k}}) = 1 - P(T \le t_k) = 1 - F_T(t_k)$

experiment stop time

from pg 11 of tables we have $F_T(t_k) = 1 - e^{-\lambda t_k}$

$\Rightarrow 1 - (1 - e^{-\lambda t_k}) = e^{-\lambda t_k}$ H.P

b) $L(\lambda) = \prod_{i=1}^{k} f(t_i) \prod_{j=k+1}^{n} P(T > t_k) \longrightarrow$ n-k policies still active

$= \prod_{i=1}^{k} \left(\lambda e^{-\lambda t_i}\right) \prod_{j=k+1}^{n} e^{-\lambda t_k}$ ...... from part (a)

$L(\lambda) = \lambda^k e^{-\lambda \sum t_i} e^{-(n-k)\lambda t_k} \longrightarrow$ information ceases

$\log L(\lambda) = k \log \lambda - \lambda \sum_{i=1}^{k} t_i - (n-k)\lambda t_k$

taking derivative w.r.t λ

$L'(\lambda) = \frac{k}{\lambda} - \sum t_i - (n-k)t_k \longrightarrow 1$

equating $\frac{d\,l(\lambda)}{d\lambda} = 0$ we get

$$0 = \frac{k}{\lambda} - \Sigma t_i - (n-k)t_k$$

$$\frac{k}{\lambda} = \Sigma t_i + (n-k)t_k \Rightarrow \boxed{\hat{\lambda} = \frac{k}{\Sigma t_i + (n-k)t_k}}$$

derivating eq ① w r t λ again

$$\frac{d^2 L(\lambda)}{d\lambda^2} = -\frac{k}{\lambda^2} < 0$$

c) $n = 20$ $k = 5$ $t_k = 21.54$ (last expiry)

$$\sum_{i=1}^{k=5} t_i = 54.82$$

$$\hat{\lambda} = \frac{k}{\sum t_i + (n-k) t_k} = \frac{5}{54.82 + 15 \times 21.54} = 0.0132$$

(ii a) n policies which are independent and each of these have prob p of expiring at time t_0

$Bin(n, p)$ so we have to find p

$$p = P(T \leq t_0) = 1 - e^{-\lambda t_0}$$

and policies will remain active with prob $1 - p$

$$K \sim bin(n, \underline{1 - e^{-\lambda t_0}})$$

PQ 8 Sept 09 Q5

(i) $E(X) = \int_0^\theta x f(x)\,dx = \int_0^\theta 2x^2\theta^{-2}\,dx$

$E(X) = \frac{2}{\theta^2}\int_0^\theta x^2\,dx = \frac{2\theta^{\cancel{3}}}{3\cancel{\theta^2}} = \frac{2\theta}{3}$ → biased estimator

$\boxed{S = \frac{3}{2}\bar{X}}$ → Estimator

$E(S) = \frac{3}{2}E[X] = \frac{\cancel{3}}{\cancel{2}} \times \frac{\cancel{2}}{\cancel{3}} \times \theta$ ⇒ unbiased estimator

PQ5 Sept 18 Q10

(i) Solved Verbally

(ii) $X_1 + x_2 + x_3 = n$ claims

$x_1 = rej$ @ 1st

$x_2 = rej$ @ 2nd

$x_3 = pass$ @ 2nd

$$L(\theta) = (1-\theta)^{x_1} \left[\theta(1-\theta)\right]^{x_2} (\theta^2)^{x_3}$$

$$= (1-\theta)^{x_1 + x_2} \theta^{x_2 + 2x_3}$$

$$\log L(\theta) = (x_1 + x_2) \log(1-\theta) + (x_2 + 2x_3)(\log \theta)$$

$$\frac{d}{d\theta}\log L(\theta) = -\frac{(x_1+x_2)}{(1-\theta)} + \frac{x_2+2x_3}{\theta}$$

equating to zero

$$\frac{x_2+2x_3}{\theta} = \frac{x_1+x_2}{1-\theta} \Rightarrow \theta(x_1+x_2) = (1-\theta)(x_2+2x_3)$$

$$= \theta(x_1+x_2+x_2+2x_3) = x_2+2x_3$$

$$\Rightarrow \hat{\theta} = \frac{x_2+2x_3}{x_1+2x_2+2x_3}$$

$$\frac{d^2 \log L(\theta)}{d\theta^2} = \frac{d}{d\theta}\left[\frac{(x_2+2x_3)}{\theta} - \frac{(x_1+x_2)}{1-\theta}\right]$$

$$= -\frac{(x_2+2x_3)}{\theta^2} - \frac{1\,(x_1+x_2)}{(1-\theta)^2}$$

$$-E\left[\frac{1}{\frac{d^2 \log L(\theta)}{d\theta^2}}\right] = CRLB$$

$$E\left[-\frac{(x_1+x_2)}{(1-\theta)^2} - \frac{(x_2+2x_3)}{\theta^2} \right]$$

$$\frac{-n(1-\theta) - n\theta(1-\theta)}{(1-\theta)^2} - \frac{n\theta(1-\theta) - 2n\theta^2}{\theta^2}$$

$$= -n - \frac{n\theta}{1-\theta} - \frac{n(1-\theta) - 2n\theta}{\theta}$$

$$\Rightarrow -\frac{n(1+\theta)}{1-\theta} - \frac{n[1-\theta+2\theta]}{\theta} = -\frac{n(1+\theta)}{1-\theta} - \frac{n(1+\theta)}{\theta}$$

$$-n(1+\theta)\left[\frac{1}{1-\theta} + \frac{1}{\theta}\right]$$

$$-n(1+\theta)\left[\frac{\cancel{\theta}+1-\cancel{\theta}}{\theta(1-\theta)}\right] \Rightarrow -\frac{n(1+\theta)}{\theta(1-\theta)} < 0$$

$$\frac{1}{-E\left[\frac{\partial^2 \log L(\theta)}{\partial \theta^2}\right]} = \frac{1}{-\left[-\frac{n(1+\theta)}{\theta(1-\theta)}\right]}$$

$$\frac{\theta(1-\theta)}{n(1+\theta)} = CRLB$$

P03 Unit 1 Apr 08 Q8

$A \sim N(0, 9 \times 100)$ $A \sim N(0, 900)$

σ^2 $n = 100$

$P(A > 15) = 1 - P(A \leq 15)$

$\Rightarrow 1 - P\left(Z \leq \frac{15 - 0}{\sqrt{900}}\right) = 1 - P\left(Z \leq \frac{15}{30}\right)$

$1 - P(Z \leq 0.5) \Rightarrow$ Answer

$$P(|Y| \geq 15)$$

$$= P(Y < -15) + P(Y > +15)$$

$$= P(Z < -0.5) + P(Z > 0.5)$$

$$= P(Z > 0.5) + P(Z > 0.5)$$

$$= 2(P(Z > 0.5))$$

$$= 2(1 - P(Z < 0.5)) = 2(1 - 0.69146)$$

$$= 0.61708$$