Answer 1: Easy

- EMH states that market fully reflects all available information and the implication is therefore that investors are not able to make “excess” returns (rather than any returns at all!).
- 3 forms of EMH defining what type of information is available: weak for historical price information, semi-strong for all public information and strong for all information.
- Although illegal, insider information appears to enable investors to make money. Reasonable to conclude the other way round as studies of directors’ share dealings suggest that, even with inside information, it is difficult to out-perform.
- Difficult to define publicly available information – might be that some very difficult-to-obtain information enables profits but at a high cost of obtaining the information.
- Investors taking higher risks may earn higher returns – this does not contradict the EMH.
- EMH does not specify how information is priced, so very difficult to test.
- Conflicting empirical evidence from supporters and detractors.
- Difficult to determine when, precisely, information arrives.

Answer 2: Difficult

(i) Let J be the return to the investor in the junior loan.

$J = 54$, with probability 0.75

$= 0$ with probability $0.25 * 0.5$

$= 50 * U$ with probability $0.25 * 0.5$, where U is uniform over (0,1)

$E[J] = 0.75 * 54 + 0.25 * 0.5 * 0 + 0.25 * 0.5 * 0.5 * 50 = 43.625$

$E[J^2] = 0.75 * 54^2 + 0 + 0.25 * 0.5 * 50^2 * E[U^2]$
$= 0.75 * 54^2 + 312.5 * (0.25 + 0.083) = 2291$

$\text{Var}[J] = 2291 - 43.625^2 = 388$

S = return to investor in senior loan

$S = 53$ with prob 0.75

$= 50$ with prob $0.25 * 0.5$

$= 50 * U$ with prob $0.25 * 0.5$

$E[S] = 0.75 * 53 + 0.125 * 50 + 0.125 * 50 * 0.5 = 49.125$

$E[S^2] = 0.75 * 53^2 + 0.125 * 50^2 + 0.125 * 50^2/3 = 2523$
$\text{Var}[S] = 2523 - 49.125^2 = 110$

(ii) $\Pr(J < 50) = 0.25$
$\Pr(S < 50) = 0.125$

Answer 3: easy

(i) $E[R] = 0.8 * 0\% + 0.2 * 30\% = 6\%$
$E[R^2] = 0.8 * 10\%^2 + 0.2*(30\%^2 + 10\%^2) = 0.028$

(ii) $\text{Var}(R) = 0.028 - 0.06^2 = 0.0244 = 0.1562^2$

$\text{Prob}(R < 0) = 0.8 * N(0; 0, 10\%) + 0.2 * N(0; 30\%, 10\%) = 0.8 * 0.5 + 0.2 * 0.00135 = 0.4 + 0.00027 = 0.40027$

$\text{Prob}(S < 0\%) = N(0; 6\%, 15.62\%) = 0.3504$

$\text{Prob}(R < -10\%) = 0.8 * N(-10\%; 0, 10\%) + 0.2 * N(-10\%; 30\%, 10\%) = 0.8 * 0.1587 + 0.2 * 0 = 0.1269$

$\text{Prob(S} < 10\%) = N(-10\%; 6\%, 15.62\%) = 0.1528$

(iii) Variance suggests risks are the same

Benchmark at 0% suggests R riskier than S – "weight" of probability around 0% with R makes R look riskier than S

Benchmark at –10% suggests S riskier than R – overall wider "spread" of S dominates at more extreme risk levels

(Note: candidate answers may differ slightly because approximations required in standard normal lookups from tables.)

Answer 4: (easy)

(i) Strong form EMH: market prices incorporate all information, both publicly available and also that available only to insiders.

Semi-strong form EMH: market prices incorporate all publicly available information.

Weak form EMH: the market price of an investment incorporates all information contained in the price history of that investment.

(ii) Any reasonable comments :- the market was expecting more and reacted efficiently on the release of insider information. This does suggest that Strong form EMH doesn't hold. It doesn't seem to contradict weak or semi-strong EMH. However, the price fall could be an over-reaction which would contradict the semi-strong form.

Answer 5: Difficult

(i) $VaR_\alpha(X) = -t$ where $\mathbb{P}(X < t) = \alpha$.

(ii) Following the hint in the question:

$$\alpha = \mathbb{P}(X < -VaR_\alpha)$$

$$= \mathbb{P}\left(\frac{X-\mu}{\sigma} < \frac{-VaR_\alpha - \mu}{\sigma}\right)$$

$$= \mathbb{P}\left(Z < \frac{-VaR_\alpha - \mu}{\sigma}\right)$$

where Z is a standard Normal random variable.

Therefore $= \Phi\left(\frac{-VaR_\alpha - \mu}{\sigma}\right)$

$$\Rightarrow \Phi^{-1}(\alpha) = \frac{-VaR_\alpha - \mu}{\sigma},$$

and so $VaR_\alpha = -(\mu + \sigma\Phi^{-1}(\alpha))$.

.

(iii) As the loss distribution is continuous, we have

$$TailVaR_{\alpha} = \mathbb{E}\left(-X \middle| X < -VaR_{\alpha}\right)$$

$$= -\frac{1}{\alpha} \int_{-\infty}^{-VaR_{\alpha}} x\varphi_{\mu,\sigma}(x)\,dx$$

$$= -\mu - \frac{\sigma}{\alpha} \int_{-\infty}^{-\Phi^{-1}(\alpha)} x\varphi_{0,1}(x)\,dx$$

$$= -\mu - \frac{\sigma}{\alpha}\left[-\varphi_{0,1}(x)\right]_{-\infty}^{-\Phi^{-1}(\alpha)}$$

$$= \frac{\sigma}{\alpha}\varphi\left(\Phi^{-1}(\alpha)\right) - \mu$$

(iv) $VaR = -£350m\left[10\% - 25\% \times 2.32635\right] = £168.56m$

$$TailVaR = -£350m\left[10\% - \frac{25\%}{1\%}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\times 2.32635^2}\right] = £198.21m$$

There were typographical errors in the question which should have defined $TailVaR_{\alpha} = \mathrm{E}(-X|X < -VaR_{\alpha})$. *Generous consideration was given to all scripts containing any reasonable attempt in the marking of this question.*

Answer 6: Easy

(i) $U'(w) > 0$.

Answer 7: Easy

Variance of return

Variance is mathematically tractable.

Variance fits neatly with a mean-variance portfolio construction framework.

Variance is a symmetric measure of risk. The problem of investors is really the downside part of the distribution.

Credit risky bonds have an asymmetric return distribution and as defaults are often co-dependent on economic downturns portfolios can have fat tails.

Neither skewness or kurtosis of returns is captured by a variance measure.

Downside semi-variance of return

Semi-variance is not easy to handle mathematically and it takes no account of variability above the mean.

Furthermore if returns on assets are symmetrically distributed semi-variance is proportional to variance.

As with variance of return, semi-variance does not capture skewness or kurtosis.

It takes into account the risk of lower returns.

It can be decomposed into systematic and non-systematic risk contributions.

Shortfall probability

The choice of benchmark level is arbitrary.

For a portfolio of bonds, the shortfall probability will not give any information on:

• upside returns above the benchmark level • nor the potential downside of returns when the benchmark level is exceeded.

It gives an indication of the possibility of loss below a certain level.

It allows a manager to manage risk where returns are not normally distributed.

Value at Risk (VaR)

VaR generalises the likelihood of underperformance by providing a statistical measure of downside risk.

Portfolios exposed to credit risk, systematic bias or derivatives may exhibit nonnormal distributions.

The usefulness of VaR in these situations depends on modelling skewed or fat-tailed distributions of returns.

The further one gets out into the “tails” of the distributions, the more lacking the data and, hence, the more arbitrary the choice of the underlying probability distribution becomes.

Tail Value at Risk (TailVaR)

Relative to VaR, TailVaR provides much more information on how bad returns can be when the benchmark level is exceeded.

It has the same modelling issues as VaR in terms of sparse data, but captures more information on tail of the non-normal distribution.

Answer 8: easy

(i) (a) The expected utility theorem states that a function, U(w) can be constructed representing an investor’s utility of wealth, w, at some future date. Decisions are made on the basis of maximising the expected value of utility under the investor’s particular beliefs about the probability of different outcomes.

(b) The expected utility theorem can be derived formally from the following four axioms.

1. Comparability

An investor can state a preference between all available certain outcomes.

2. Transitivity

If A is preferred to B and B is preferred to C, then A is preferred to C.

3. Independence

If an investor is indifferent between two certain outcomes, A and B, then he is also indifferent between the following two gambles:

(a) A with probability p and C with probability (1 − p); and (b) B with probability p and C with probability (1 − p).

4. Certainty equivalence

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability (1 − p).

B is known as the certainty equivalent of the above gamble.

(ii) It is usually assumed that people prefer more wealth to less. This is known as the principle of non-satiation and can be expressed as: $U'(w)>0$ or U is strictly increasing.

Attitudes to risk can also be expressed in terms of the properties of utility functions.

A risk averse investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. The utility function condition is $U''(w) < 0$ or U is strictly concave.

(iii) The absolute risk aversion A is given by:

$$A(w) = \frac{-U''(w)}{U'(w)}.$$

Which for the utility function given can be calculated by taking derivatives as, $\frac{-2b}{1+2bw}$.

Now, given the condition $A(1) = 0.25$ yields $b = -0.1$.

Non-satiation means $U'(w) > 0 \iff 1+2bw > 0 \iff -\infty < w < 5$.

Answer 9: Easy

(i) Strong form EMH: market prices incorporate all information, both publicly available and also that available only to insiders.

Semi-strong form EMH: market prices incorporate all publicly available information.

Weak form EMH: the market price of an investment incorporates all information contained in the price history of that investment.

(ii) Scenario 1: The first event tells us nothing about the EMH-assuming this earthquake was not predictable, its happening could not have been discounted in market prices. A quick adjustment of prices in response to a news announcement suggests evidence for the semi-strong form (and by implication the weak form) EMH.

However, although the price drop was quick, we have no idea how accurate it was. It is possible that the market has over or under reacted to the bad news and will correct itself later. If this is the case, then it suggests markets are not efficient.

Some earthquake specialists (insiders) may have known about the earthquake shortly in advance but there is no mention of price movements before the earthquake, perhaps this suggests the market is also strong form efficient.

Scenario 2: The second event strongly contradicts the strong-form EMH. Insiders are privy to all information about the merger talks and therefore there shouldn’t be a sudden reaction.

Indeed, given the public nature of the negotiations, this seems even to contradict the semi-strong form (and by implication the strong form) of the EMH although perhaps markets were pricing in a significant probability of the merger failing or overreacting to the benefits and then correcting themselves.

Answer 10:

(ii) Non-satiated: $U'(w) > 0$
Risk neutral: $U''(w) = 0$

(iii) $R(w) = -wU''(w) / U'(w)$
$= 1 - \gamma$

(iv) $R'(w) = 0$ so the relative risk aversion is constant; "iso-elastic" is also acceptable.

Answer 11:

(i) VaR $= -t$ where $P(X < t) = 5\%$

(ii) Expected shortfall $= E[\max(2\% - X, 0)]$

$$= \int_{-\infty}^{2\%} (2\% - x) f(x) dx$$

(iii) $P(X < t) = 5\%$ where $x \sim N(5, 100)$
$\Leftrightarrow P(Z \le (t-5) / 10) = 5\%$
$\Leftrightarrow (t-5) / 10 = -1.645$
$\Leftrightarrow t = -11.45\%$

Therefore the 5% VaR is $-t = 11.45\%$.

(iv) VaR does not illustrate the size of the loss in the tail of the distribution, only the likelihood.

The usefulness of VaR may be limited by a lack of data to determine the tail of the distribution.

Answer 12:

(i) $A(w) = -U''(w) / U'(w) = -2c / (b+2cw)$
$R(w) = -wU''(w) / U'(w) = -2cw/(b + 2cw)$

(ii) For a gamble with an equal size gain or loss, the requirement that $p \ge 0.55$ implies that the investor is risk averse. (Alternatively, they have increasing absolute and relative risk aversion.)

(iii) With w =100, the (certain) utility if the gamble is rejected is:

(1) $610 = a + 100b + 10{,}000c$

whereas the expected utility if the gamble is accepted with $p = 0.55$ is:

$U(100) = 0.55 * U(120) + 0.45 * U(80)$
$\Rightarrow 610 = 0.55 * (a + 120b + 14{,}400c) + 0.45 * (a + 80b + 6{,}400c)$
(2) $\Rightarrow 610 = a + 102b + 10{,}800c$

With w =120, the (certain) utility if the gamble is rejected is:

(3) $U(120) = a + 120\text{b} + 14{,}400c$

whereas the expected utility if the gamble is accepted with $p = 0.5625$ is:

$U(120) = 0.5625 * U(140) + 0.4375 * U(100)$
$\Rightarrow U(120) = 0.5625 * (a + 140b + 19{,}600c) + 0.4375 * (a + 100b + 10{,}000c)$
(4) $\Rightarrow U(120) = 0.4375 * U(100) + 0.5625 * U(140) = a + 122.5b + 15{,}400c$

Solving these gives:

$a = (17{,}080 - 25 * U(120)) / 3$
$b = (U(120) - 610) / 9$
$c = -(U(120) - 610) / 3{,}600$

(iv) $U'(w) = b + 2cw$
$U'(w) = 0 \Rightarrow w = -b / 2c$
$b = -400c$ (from above) $\Rightarrow w = £200$

Answer 13: easy

(i) $U'(w) = 1/w$

$U''(w) = -1/w^2$

Absolute risk aversion $= A(w) = -U''(w)/U'(w)$

$= 1/w$

$A'(w) = -1/w^2$

Relative risk aversion $= R(w) = -wU''(w)/U'(w)$

$= 1$

$R'(w) = 0$

(ii) R'(w) =0 thus the log utility function exhibits constant relative risk aversion.

This is consistent with an investor who keeps a constant proportion of wealth invested in risky assets as she gets richer.

(iii) Wealth after the uncertain event will be either:

$$100 \times (1.3a + (1 - a)) = 100 + 30a \text{ with probability } 0.75$$

or:

$$100 \times (0.4a + (1 - a)) = 100 - 60a \text{ with probability } 0.25.$$

Thus expected utility of wealth is:

$$0.75 \times \ln(100 + 30a) + 0.25 \times \ln(100 - 60a).$$

(iv) Differentiate with respect to a:

$$30 \times 0.75/(100 + 30a) - 60 \times 0.25/(100 - 60a).$$

Set equal to zero:

$30 \times 0.75 / (100 + 30a) - 60 \times 0.25 / (100 - 60a) = 0$
$30 \times 0.75 / (100 + 30a) = 60 \times 0.25 / (100 - 60a)$
$30 \times 0.75 \times (100 - 60a) = 60 \times 0.25 \times (100 + 30a)$
$22.5 \times (100 - 60a) = 15 \times (100 + 30a)$
$2250 - 1350a = 1500 + 450a$
$750 = 1800a$
$a = 0.4167$

Check for maximum:

Differentiate with respect to a again:

$$-30^2 \times 0.75/(100 + 30a)^2 - 60^2 \times 0.25/(100 - 60a)^2.$$

This must be negative because of the square terms, hence this is a local maximum.

Answer 14:

(i) Variance of return is defined as:

$$\int_{-\infty}^{\infty} (\mu - x)^2 f(x)dx,$$

where μ is the mean return at the end of the chosen period.

(ii) Shortfall probability $= \int_{-\infty}^{L} f(x)dx$.

(iii) The shortfall probability required is the probability that the return is lower than $480/500 - 1 = -4\%$ i.e. $P(N(6\%, 23\%) \leq 4\%)$
$= P(Z \leq (-4\% - 6\%)/\sqrt{(23\%)})$
$= P(Z \leq -0.20851)$
$= 0.417$

(iv) (a) This may imply that the investor has a quadratic utility function.

(b) This corresponds to a utility function which has a discontinuity at the minimum required return.

Answer 15: Easy

(ii) Tests need to make assumptions (which may be invalid) such as normality of returns or stationarity.

Transaction costs may prevent the exploitation of anomalies, so that the EMH might hold net of transaction costs.

Allowance for risk: the EMH does not preclude higher returns as a reward for risk; however the EMH does not tell us how to price such risks.

Testing the strong form EMH is problematic as it requires access to information that is not in the public domain.

It can be difficult to define “public information” or to determine exactly when information becomes public.

It is impossible to test all of the possible trading rules that might be used by technical analysts.

The assumptions made about how security prices should react to new information may be invalid.

Answer 16:

(i) $\mathrm{VaR}(X) = -t$ where $P(X < t) = p$.

(ii) $\int_{-\infty}^{\mu} (\mu - x)^2 f(x)dx$ where μ is the mean return at the end of the chosen period.

[Or equivalently $\int_{-\infty}^{\mu} (x-\mu)^2 f(x)dx$ *]*

(iii) For:

Most investors do not dislike uncertainty of returns as such; rather they dislike the possibility of low returns. One measure that seeks to quantify this view is downside semi-variance.

Against:

Semi-variance is not easy to handle mathematically.

Semi-variance takes no account of variability above the mean.

Furthermore if returns on assets are symmetrically distributed semi-variance is proportional to variance, so it gives no extra information. Semi-variance measures downside relative to the mean rather than another benchmark that might be more relevant to the investor.

(iv)

$P(X = 0) = (8^0 e^{-8})/0! = 0.00034$
$P(X = 1) = (8^1 e^{-8})/1! = 0.00268$
$P(X = 2) = (8^2 e^{-8})/2! = 0.01073$
$P(X = 3) = (8^3 e^{-8})/3! = 0.02863$
$P(X = 4) = (8^4 e^{-8})/4! = 0.05725$
So $P(X \leq 4) = 0.00034 + 0.00268 + 0.01073 + 0.02863 + 0.05725 = 0.09963$

Alternatively, directly from the Formulae & Tables: $P(X \leq 4) = 0.09963$ [

$P(X = 5) = (8^4 e^{-8})/5! = 0.09160$
So $P(X \leq 5) = 0.191236$ *(or directly from the Formulae & Tables)* [

So the 10% VaR level is 5 (or –5) apples.

(v) $P(X = 0) \times (5 - 0) = 0.002$
$P(X = 1) \times (5 - 1) = 0.011$
$P(X = 2) \times (5 - 2) = 0.032$
$P(X = 3) \times (5 - 3) = 0.057$
$P(X = 4) \times (5 - 4) = 0.057$

Summing the above, we get 0.159

So the expected shortfall below 5 apples is 0.159 apples

Answer 17:

(ii) $U'(w) = 1 + 2dw$, and [½]

$U''(w) = 2d$. [½]

Because the investor is risk averse, we must have $U''(w) < 0$ *(alternatively to satisfy the condition of diminishing marginal utility of wealth (risk aversion))* [½]

So we must have $d < 0$. [½]

(iii) The condition of non-satiation requires $U'(w) > 0$. [½]

Hence $1 + 2dw > 0$ and $w < -1/(2d)$ [½]

So the quadratic utility function can only satisfy the condition of non-satiation over a limited range of w:

Specifically $-\infty < w < -1/(2d)$ [1]

(iv) $1{,}000 = -1/2d$ [½]

$\Rightarrow d = -1/2000 = -0.0005$ [½]

(v) $U(250) = 250 - 0.0005 \times 250^2 = 218.75$ [1]

$E[U(\text{exchange})] = 0.5 \times U(600) + 0.5 \times U(0)$ [½]

$= 0.5 \times (600 - 0.0005 \times 600^2) = 210$ [½]

So the investor should not accept the opportunity to exchange… [½]

… because the expected utility of the exchange opportunity is lower than that of the prize. [½]

Answer 18:

i)

a. $U'(w) = w^{-0.5} > 0$ for $w > 0$
b. $U''(w) = -0.5w^{-1.5} < 0$ for $w > 0$

ii) $R(w) = w*(-U''(w)/U'(w)) = 0.5$

iii) E[U] = 0.6U(1.69a) + 0.1U(6.25b) +0.3U(0)
= 0.6*2*((1.69a)^0.5-1) + 0.1*2*((6.25b)^0.5-1) – 0.6
= 1.2*(1.3a^0.5-1) + 0.2*(2.5b^0.5-1) – 0.6
= 1.56a^0.5 + 0.5b^0.5 – 1.4 – 0.6
= 1.56a^0.5 + 0.5(1000-a)^0.5 –2
dE[U]/da = 0.78a^-0.5 – 0.25(1000-a)^-0.5
Setting dE[U]/da = 0 gives
0.78a^-0.5 = 0.25(1000-a)^-0.5
=> 0.78 = 0.25a^0.5(1000-a)^-0.5
=> 3.12 = (a/(1000-a))^0.5
Squaring both sides
=> 9.7344 = a/(1000-a)
=> 9,734.4 = 10.7344a or 8.7344a
=> a = £906.8 or £1,114.50
Rejecting the figure >£1,000 gives
a = £906.80 and b=£93.20
Checking the second derivative
d^2E[U]/da^2 = -0.39a^-1.5 – 0.25(1000-a)^-1.5 < 0
hence this is a maximum

iv) E[U] = 49.801 [1]
[Note to markers: assign [1] mark to answers containing the expected wealth]

v) U(1000) = 61.2456 [1]
So the maximum expected utility of wealth is less than the current utility of wealth. [½]
This is because the odds offered pay out less than would be required based on the investor's estimated probabilities of each horse winning. [½]
Based on expected utility, the investor would be better off not betting at all. [½]
There may be other horses in the race where this position is reversed. [½]

Answer 19: easy

(i) Weak Form EMH

The market price of an investment incorporates all information contained in the price history of that investment. Knowledge of a stock's price history cannot produce excess performance as this information is already incorporated in the market price. This form, if true, means that technical analysis (or chartism) techniques (i.e. analysing charts of prices and spotting patterns) will not produce excess performance.

Semi-Strong Form EMH

The market price of an investment incorporates all publicly available information. Knowledge of any public information cannot produce excess performance, as this information is already incorporated in the market price. This form, if true, means that fundamental analysis techniques (i.e. analysing accounting statements and other pieces of financial information) will not produce excess performance.

Strong Form EMH

The market price of an investment incorporates all information, both publicly available and that available only to insiders. Knowledge available only to insiders cannot produce excess performance as this information is already incorporated in market prices.

(ii) (a) Some of the effects found by studies can be classified as overreaction to events, for example:
The market appears to overreact to past performance. Past winners tend to be future losers and vice versa.

Certain accounting ratios appear to have predictive powers e.g. companies with high earnings to price, cashflow to price and book value to market value (generally poor past performers) tend to have high future returns. Again, this is an example of the market apparently overreacting to past growth.

(b) There are also well-documented examples of under-reaction to events:

Firms coming to the market

Evidence from a number of major financial markets including the UK and the US appears to support the idea that stocks coming to the market by Initial Public Offerings and Seasoned Equity Offerings have poor subsequent long-term performance.

Shiller's analysis

Shiller found strong evidence that the observed level of volatility in S&P 500 stock index contradicted the EMH as such volatility was not in line with the subsequent fluctuations in the dividends. Also, if markets are efficient, broad movements in the perfect foresight price should be correlated with moves in the actual price as both react to the same news.

Stock prices continuing to respond to earnings announcements up to a year after their announcement This is an example of under-reaction to information which is slowly corrected.

Abnormal excess returns for both the parent and subsidiary firms following a demerger. This is another example of the market being slow to recognise the benefits of an event.

Abnormal negative returns following mergers (agreed takeovers leading to the poorest subsequent returns). The market appears to overestimate the benefits from mergers……and the stock price slowly reacts as the optimistic view is proved to be wrong.

Answer 20:

(i) (a) $1{,}000{,}000 \times 1.01 = 1{,}010{,}000$

(b) $200{,}000^2 = 4 \times 10^{10}$

(c) $0.5 \times 200{,}000^2 = 2 \times 10^{10}$

(d) Let the one-day portfolio return be denoted x :
$P(1{,}000{,}000\ (1 + x) < 1{,}000{,}000) = P(x < 0)$
$= P((x - 1\%)/20\% < -0.05) = P(Z < -0.05) = 0.48006$

(ii) $N\!-\!1(0.01) = -2.3263$
$£1{,}000{,}000 \times 1.01 + £1{,}000{,}000 \times 20\% \times -2.3263 = £544{,}740.$
So the 99% Value at Risk is $£1{,}010{,}000 - £544{,}740 = £465{,}260$

(iii) P(not seeing a 99% VaR event in n days) $= 0.99^n$.
So we want $0.99^n > 0.5$
So $n \log 0.99 > \log 0.5$, so $n \geq 69$ days

Answer 21:

(i)
(a)
Shortfall probability $= \int_{-\infty}^{L} f(x)dx$

(b)
VaR(X) = -t where P(X < t) = p
Or also accept P(X < t) = 1 - p

(c)
$\int_{-\infty}^{\mu} (\mu - x)^2 f(x)dx$

(ii)
(a)
Bank account = £9,000 x 1.1 = £9,900 hence 100% shortfall prob. [½
Shares = P(X<(10/9-1)) where X~N(0.15,0.15^2) = 40% [1]
Gamble = shortfall if not successful hence 35% shortfall prob. [½

(b)
Bank account = guaranteed value so VaR = £9,900 [1]
Shares = P(X<t) = 0.25 where X~N(0.15,0.15^2) => t=0.0488 => VaR = 9,000*(1+0.0488) = £9,439 [1]
Gamble = 35% chance of ending with nothing hence VaR = £9,000 [1]

(iii) The two risk measures are not conclusive and each suggests that a different investment would be best.

In reality the bank account delivers nearly enough money with no risk so the student might be best to either invest wholly in the bank account and wait a little longer to buy the car or, if allowed to split the investment, invest mostly in the bank account and a little in the shares or the gamble.

The student could also seek out other investments with a different risk/return profile. The shortfall probability as it assumes all the student cares about is reaching £10k over one year. The student might also consider the size of any shortfall or surplus. If the student only needs £10,000 then it makes no sense to invest more than £5,000 in the gamble.

Answer 22:

(i)
(a) $U'(w) > 0$
(b) $U''(w) < 0$
(c) $A'(w) > 0$ where $A(w) = -U''(w) / U'(w)$
(d) $R'(w) < 0$ where $R(w) = -wU''(w) / U'(w)$

(ii)
$U'(w) = 1 + 2dw$
$U''(w) = 2d$ [0.5]
Hence $A(w) = -2d / (1 + 2dw)$

(iii)
$A'(w) = 4d^2 / (1 + 2dw)^2 > 0$ hence increasing

(iv)
Current utility = $250 - 0.001 \times 250^2 = 187.5$ [
Expected utility = $P(X=0) \times U(250 + 2p) + P(X=1) \times U(250 + 2p - 100) + P(X=2) \times U(250 +$
$2p - 2 \times 100)$ [½
$= 0.9^2 \times (250 + 2p - 0.001 \times (250 + 2p)^2) + 2 \times 0.9 \times 0.1 \times (150 + 2p - 0.001 \times (150 + 2p)^2) +$
$0.1^2 \times (50 + 2p - 0.001 \times (50 + 2p)^2)$ [½
$= 0.81 \times (187.5 + p - 0.004p^2) + 0.18 \times (127.5 + 1.4p - 0.004p^2) + 0.01 \times (47.5 + 1.8p -$
$0.004p^2)$ [½
$= (175.3 + 1.08p - 0.004p^2)$ [½

So equating the two sides:
$187.5 = 175.3 + 1.08p - 0.004p^2$ [½
So $0 = -12.2 + 1.08p - 0.004p^2$ [½
We can reject the root at £258.19 as clearly too large. [½
So p = £11.81 [½

(v)
The premium of £11.81 is higher than the expected claim of $0.1 \times £100 = £10$

Initially this does not appear attractive to customers
However, customers buy insurance to reduce risk and increase certainty of cost
So this might still be attractive
If we knew the customer's utility function and initial wealth we could determine whether this is attractive.

Answer 23:

(i)(a)

$X = 720/800 - 1 = -10\%$

$P(X<-10\%) = P(Z<(-10\% - 7\%)/5.5\%) = P(Z<-3.09) = 0.1\%$

(b)

$P(Z<(t - 7\%)/5.5\%) = 0.005$

$(t - 7\%)/5.5\% = -2.5758$

$t = -7.1669\%$

$800 * (1-7.1669\%) = \$742.66$

(ii)

$P(X\leq -7.1\%) = P(Z\leq -2.56)$

$=0.00518$

$P(X>7\%) = 0.5$ (as 7% is the mean)

$P(-7.1\%<X\leq 7\%) = 1-0.5-0.00518 = 0.49482$

Expected pay out = 730*0.00518 + 750*0.49482 + 962*0.5 = \$855.90

(iii)(a)

0%

(b)

Probability pay out is $\leq$ 730 is 0.52% therefore the 99.5% VaR is \$730

(iv) The expected return from investing in the index is 800*1.07 = $856. So the expected returns are very similar for each investment. Based on the expected shortfall below $720 the derivative is less risky as there is no possibility of this . If the investor has a utility function with a discontinuity at the minimum required return then he may base his decision on this measure. The 99.5% VaR is higher (i.e. a greater loss) for the derivative, so based on this measure the investor may prefer to invest in the stock index . The pay off on the derivative is significantly higher than the index when the return is slightly above the mean, so the investor may prefer this.

Answer 24:

(i)(a)
Asset A
Variance = 2^2 * npq = 5.76%%
Asset B
Variance = 1.5^2 * σ^2 = 9%%

(b)
Asset A
P(2X<3) = P(X<1.5) = P(X≤1) = 0.2333
Asset B
P(1.5Y<3) = P(Y<2) = P(Z<-0.6) = 0.27425

(ii)
Mean returns: Asset A = 2*6*0.4 = 4.8%, Asset B = 1.5*3.2 = 4.8%. [1]
Both assets have same mean therefore investor will choose the asset with the lowest variance. [1]
The investor will choose Asset A. [1]
The quadratic utility function also implies variance as the risk measure [1]

(iii)(a) If the assets are independent then a combination of both assets will give a lower variance than either asset on its own. But with the same expected return. So the investor would prefer to invest partially in each asset.

(b) If the assets exhibit correlation and short selling is not allowed then the variance of a combined portfolio will be higher than if they were uncorrelated. But it will still be lower than investing in a single asset. So the investor would still prefer to invest partially in each asset. Or, if short selling is allowed, the investor could short one asset in order to achieve a lower portfolio variance.

Answer 25:

(i)
For a utility function to be valid, it must respect the fact that the investor is non-satiated
This is equivalent to requiring that $U'(w)>0$
Substituting the given form of (w) we find the following:
$U'(w) = 1+2dw > 0$
$w < -1 / 2d$

(ii)
For a utility function to be valid, it must also respect the investor being risk-averse
This is equivalent to requiring that $U''(w)<0$
Substituting the given form of (w) we find the following:
$U''(w) = 2d < 0$, which requires that d is negative

(iii)

If the investor buys 7 boxes of vegetables, they have £30 remaining in cash

For each of the payoffs, their final wealth will be:

Payoff	Final Wealth
30	210 + 30 = 240
12	84 + 30 = 114
10	70 + 30 = 100
0.5	3.5 + 30 = 33.50

So their expected utility is:

$(w) = 0.25(240 + 114 + 100 + 33.5) + 0.25(2402 + 1142 + 1002 + 33.52)$

$(w) = 121.875 + 20429.5625d = 50$

$d = (50 - 121.875) / 20429.5625 = -0.003518$

(iv)

$(100) = 100 - 0.003518 * 1002 = 64.82$

(v) The investor bought the vegetables despite this strategy having lower expected utility according to the utility function they chose. The investor is risk-averse, so they should make the decision with higher expected utility. So the utility function may not be appropriate for the investor, because it is not consistent with the decisions they are making or because for this decision the investor is not as risk-averse i.e. they may be risk-seeking when it comes to buying these vegetable. Also, the maximum wealth this utility function can be used with is £142. The investor can easily exceed this wealth in the highest payoff scenario

Answer 26:

(i)

At the end of one year the investment will be worth $100(x + (1 - x) * 1.5) = 150 - 50x$ with probability 0.6 [

or $100(x + (1 - x) * 0.5) = 50 + 50x$ with probability 0.4 [

Therefore, the expected utility is $0.6 * \ln(150 - 50x) + 0.4 * \ln(50 + 50x)$ [

(ii)

Differentiate expected utility: $U'(x) = -50 * \frac{0.6}{150-50x} + 50 * \frac{0.4}{50+50x}$

Set equal to zero and solve for x:

0.4(150-50x)=0.6(50+50x)

=> x=0.6

Check that this is a maximum:

$U''(x) = -2500 * \frac{0.6}{(150-50x)^2} - 2500 * \frac{0.4}{(50+50x)^2} < 0$

Therefore, the investor should invest \$60 in Asset A and \$40 in Asset B

Answer 27:

(i)(a)

For the exponential distribution, mean $= \frac{1}{\lambda} = 0.5$ [1]

For the lognormal distribution, $e^{\mu+\frac{\sigma^2}{2}} = e^{-1.04+\frac{0.833^2}{2}} = 0.500\ (3sf)$

(b)

For the exponential distribution, var $= \frac{1}{\lambda^2} = 0.25$

For the lognormal distribution,

$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right) = e^{-2.08+0.833^2}\left(e^{0.833^2}-1\right) = 0.250\ (3sf)$

(c)

For the exponential distribution, the CDF is $F(x) = 1 - e^{-2x}$

Inverting this and solving, the 99th percentile is at $x = -\frac{1}{2}\log(0.01) = 2.30259$

For the lognormal distribution:

$P(X < x) = 0.99$

$P(\log(X) < \log(x)) = 0.99$

$P\left(\frac{\log X-(-1.04)}{0.833} < \frac{\log(x)-(-1.04)}{0.833}\right) = 0.99$

$\frac{\log(x)+1.04}{0.833} = 2.3263$

$x = e^{0.833*2.3263-1.04} = 2.45\ (3sf)$

(ii) The two distributions have the same mean and variance. But different 99th percentiles. This shows that it is important to consider not just mean and variance. But also tail behaviour of a distribution.

(iii) The lognormal distribution has a higher 99th percentile, which suggests it models the tails of the security's behaviour more appropriately than the exponential distribution. The lognormal distribution has more parameters, so may be more flexible when fitting to historic data. The lognormal distribution in general is a more well-established choice in financial modelling and can lead to useful frameworks like Black-Scholes valuation. The exponential distribution does not have such a framework

Part (i) shows that the lognormal distribution has the heavier upper tail, and this might be a better fit to the heavy upper tail of security prices. Part (i) does not consider the lower tail of the distributions, but the density function for the exponential distribution is largest for the smallest values, which does not fit the observation that security prices cluster around the mean.

Answer 28:

(i)(a)
$\int_{-\infty}^{\infty}(\mu - x)^2 f_X(x)dx$, where $\mu = E(X)$

(b)
$\int_{-\infty}^{\mu}(\mu - x)^2 f_X(x)dx$

(c)
$\int_{-\infty}^{L}(L - x)\, f_X(x)\, dx$

(ii) Since different days are independent, we can use the binomial distribution

$$P(Number\ of\ Losses \geq 3) = 1 - P(Number\ of\ Losses \leq 2)$$

From the actuarial tables (or by direct calculation), this is 0.001

(iii) From part (ii), we know that the chances of this happening are very slim. This suggests that VaR has not been an effective risk measure. Because it has predicted that real-world events were highly unlikely And because this in practice has cost the trader more money than expected. This may be because the model was not appropriately fitted. e.g. the trader chose an inappropriate approach to fit the distribution to the data

Alternatively, it may be that VaR struggles to appropriately measure tail risk. Many distributions lack sufficiently 'fat tails' to model the extreme behaviour of a market crash. It is possible that the model has only been fitted to historical data and does not include an event similar to the market crash that just occurred. e.g. because the historical data is 'milder' than the event that has just occurred

Alternatively, it may be that other assumptions are inappropriate. e.g. the assumption that different days are independent e.g. using a one-day VaR did not cover a long-enough timeframe to capture the security's risk and a one-week/one-month VaR may have been better The exact model used by the trader might be overly complex. There might be a problem with the software tools used to run the VaR model VaR doesn't give a measure of how bad things could get if the level L is breached – this doesn't make it a very effective risk measure

Answer 29:

(i)

For the utility function to respect the lemurs being risk-averse and non-satiated, it must have the following properties:

$U'(w) > 0$

$U''(w) < 0$

(a)

$U'(w) = 1 + 2w$

This is smaller than 0 when $w < -0.5$

$U''(w) = 2$

Hence $U''(w) > 0$

This cannot be a valid utility function according to either measure

(b)

$U'(w) = \frac{\gamma w^{\gamma-1}}{\gamma} = w^{\gamma-1}$

This is positive because $w > 0$

$U''(w) = (\gamma - 1)w^{\gamma-2}$

This is negative because we are told that $\gamma < 1$

So this could be a valid function

(c)

$U'(w) = 1 - 4w$

$U''(w) = -4 < 0$

The second derivative is acceptable but the first is not acceptable because it is not valid at e.g $w = 1$, where the first derivative is -3 (or any other counterexample using $\frac{1}{4} \leq w \leq 3$)

So this cannot be a valid utility function

(ii)

Scenario A: expected utility is $E\big(U(w)\big) = 0.5(\ln(4) + \ln(1))$

$E\big(U(w)\big) = \ln(2)$

Scenario B: expected utility is $E\big(U(w)\big) = \ln(2.1)$

The expected utility to the lemurs is larger under scenario B

So if the utility function is appropriate, they should prefer scenario B

This is consistent with the experiment's findings

Answer 30:

(i)

$U(w) = w - 6w^2$

$U'(w) = 1 - 12w$

$U''(w) = -12$

$A(w) = \frac{-U''(w)}{U'(w)} = \frac{12}{(1-12w)}$

$A'(w) = \frac{144}{(1-12w)^2} > 0$

$R(w) = w\frac{-U''(w)}{U'(w)} = \frac{12w}{(1-12w)}$

$R'(w) = \frac{12}{(1-12w)} + \frac{144w}{(1-12w)^2} = \frac{12}{(1-12w)^2} > 0$

(ii)

Option A:

w = 0.05

E(U(w)) = 0.05-6*0.052 = 0.035

Option B:

w = 0.08 with probability 0.2, w = 0.06 with probability 0.7, w = 0.03 with probability 0.1

E(U(w)) = 0.2 * (0.08-6*0.082) + 0.7 * (0.06-6*0.062) + 0.1 * (0.03-6*0.032) = 0.03766

Option C:

w = 0.065 with probability 0.5, w = 0.06 with probability 0.5

E(U(w)) = 0.5 * (0.065-6*0.0652) + 0.5 * (0.06-6*0.062) = 0.039025

The investor is likely to choose option C as it gives the highest expected utility.

(iii)

One of the assumptions of the expected utility theorem is non-satiation.

In terms of utility functions this means U'(w)>0.

For this investor U'(w) = 1-12w. If 1-12w>0 then w<0.08333, i.e. £83,333.

If the investor had £65,000 then his wealth would exceed £83,333 under any outcome with a return over 28% (e.g. in option 2 if the return was 60% or in option 3 if the return was 30%).

Therefore this utility function could not be used to assess the preferred option.

Answer 31:

(i)

U(w) = (8^0.01 - 1) / 0.01 = 2.1012

(ii)
U'(w) = w^(-0.99)
U''(w) = -0.99w^(-1.99)
A(w) = 0.99/w
A'(w) = -0.99/w^2 <0
Therefore it exhibits declining absolute risk aversion
R(w) = 0.99
R'(w) = 0
Therefore it exhibits constant relative risk version

(iii)
Maximum P solves the following equation, where X represents the win from the lottery and w is current wealth (w=8):
E(U(w-P+X)) = U(w)
1/10,000*U(8-P+1,000) + 9,999/10,000*U(8-P) = 2.1012
P = 0.004
Therefore the maximum price he will pay is $4.

(iv)
If $\gamma>1$ then $U''(w)>0$
The function would not satisfy the principle of diminishing marginal utility of wealth.
This would suggest the individual was risk seeking (i.e. not risk averse).
Common utility theory assumes that individuals are risk averse.
If $\gamma>1$ then the utility function would exhibit increasing absolute risk aversion.
This is not plausible.

Answer 32:

(a)
$Var_A(3B\%) = 8.64\%\%$
$Var_B\ (4P\%) = 32\%\%$

(b)
$\mu_A = 3x4x0.4 = 4.8\%$
$\text{Semi-variance}_A = (4.8 - 0)^2 \times 81/625 + (4.8 - 3)^2 \times 216/625 = 4.106\%\%$
$\mu_B = 4x2 = 8\%$
$\text{Semi-variance}_B\ (4P\%) = (8 - 0)^2 \times e^{-2} + (8 - 4)^2 \times e^{-2} \times 2^1 / 1! = 12.992\%\%$

(c)
$Prob_A\ (3B < 4) = Prob_A\ (B < 1.333) = Prob_A\ (B = 0) + Prob_A\ (B = 1) = 0.4752$
$Prob_B\ (4P < 4) = Prob_B\ (P < 1) = Prob_B\ (P = 0) = 0.13534$