Answer 1:

(i) The market price of risk is $(EM - r)/\sigma M$ so $\sigma M = (EM - r)/0.1 = .02/.1 = 20\%$

(ii) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is $.2A + .3B + .5C$ and so $EM = .2EA + .3EB + .5EC$ so
$E_B = (.053 - .2 \times .04 - .5 \times .06)/.3 = 5\%$.

(iii) Assets all lie on the securities market line, so $Ei - r = \beta i(EM - r)$, where $\beta i = \mathrm{Cov}(Ri, RM)/\mathrm{Var}(RM)$.

It follows that $\beta A = .007/.02 = .35$, $\beta B = .017/.02 = .85$ and $\beta C = .027/.02 = 1.35$.

Then $\mathrm{Var}(RM) = .04$ (from part (i)) so $\mathrm{Cov}(RA, RM) = 0.014$, $\mathrm{Cov}(RB, RM) = 0.034$ and $\mathrm{Cov}(RC, RM) = 0.054$.

Answer 2:

(i) $E_M = 9\%$

(ii)

Asset number	*1*	*2*	*3*	*4*	*5*
Expected return	6%	5%	8%	13%	11%
Market capitalisation (in \$)	2.6m	3.9m	5.2m	6.5m	1.3m
Beta	5/8	1/2	7/8	1.5	5/4

(iii) $\beta_P = 19/24$

(iv) *P* does not belong to the Capital Market Line because (except in degenerate cases) portfolios on the efficient frontier consist of linear combinations of the market portfolio and the risk-free asset.

Answer 3:

(i) The key assumptions are:

(a) That investors select their portfolios on the basis of the expect return and the variance of that return over a single time horizon.

(b) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(c) Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance.

(ii) Suppose an investor can invest an any of N securities, $i = 1, \ldots, N$.
A proportion x_i is invested in security S_i. The return on the portfolio R_P is

$$R_P = \sum_{i=1}^{N} x_i R_i,$$

where R_i is the return on security i.

The expected return on the portfolio E is

$$E = \mathbb{E}[R_P] = \sum_{i=1}^{N} x_i E_i,$$

where E_i is the expected return on security i.

The variance is

$$V = Var[R_P] = \sum_{i,j=1}^{N} x_i x_j C_{ij},$$

where C_{ij} is the covariance of the returns on securities i and j and we write $C_{ii} = V_i$.

(iii) A portfolio is efficient if the investor cannot find a better one in the sense that it has both a higher expected return and a lower variance.

When there are N securities the aim is to choose x_i to minimise V subject to the constraints

$$\Sigma_i x_i = 1$$

and

$$E = E_P, \text{ say,}$$

in order to plot the minimum variance curve.

One way of solving such a minimisation problem is the method of Lagrangian multipliers.

The Lagrangian function is

$$W = V - \lambda(E - E_P) - \mu(\Sigma_i x_i - 1).$$

To find the minimum we set the partial derivatives of W with respect to all the x_i and λ and μ equal to zero. The result is a set of linear equations that can be solved.

The usual way of representing the results of the above calculations is by plotting the minimum standard deviation for each value of E_P as a curve in expected return – standard deviation $(E - \sigma)$ space. In this space, with expected return on the vertical axis, the efficient frontier is the part of the curve lying above the point of the global minimum of standard deviation.

Answer 4:

(i) Since equal market capitalisation: $w_A = 0.5$ and $w_B = 0.5$.

(ii) Let r_M denote the return of the market portfolio, r_A (resp. r_B) denote the return of asset A (resp. asset B).

Then, $V(r_M) = V(0.5r_A + 0.5r_B) = 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{ cov}(r_A,r_B)$.

$\text{Beta}_A = \text{cov}(r_A,r_M)/V(r_M)$
$= (0.5 * V(r_A) + 0.5 * \text{cov}(r_A,r_B))/ 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{ cov}(r_A,r_B)$

As $\text{Cov}(r_A,r_M) = \text{cov}(r_A, 0.5r_A+0.5r_B) = 0.5 * V(r_A) + 0.5 * \text{cov}(r_A,r_B)$

Similarly, $\text{Beta}_B = (0.5 * V(r_B) + 0.5 * \text{cov}(r_A,r_B)) / 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{ cov}(r_A,r_B)$

(iii) The equation of the Security Market line gives:

$\underline{r}_i = r_f + \text{Beta}_i\,(\underline{r}_M - r_f)$ where $\underline{r}_i$ is the expected return of asset i (for $i = A,B$).

Hence, using the numerical values, we get

$\underline{r}_A = 0.2$ and $\underline{r}_B = 0.16$

(iv) Using the separation theorem, we have:

$\underline{r}_P = w_0\ rf + w_M\,\underline{r}_M$

where w_0 is the weight of the risk-free asset in the portfolio P and w_M is the weight of the market portfolio in the portfolio P.

Moreover, there is the constraint $w_0 + w_M = 1$

Solving the system leads to:

$w_0 = -0.25$ and $w_M = 1.25$

(v) The Capital Market Line equation is:

$\underline{r}_P = r_f + \text{sigma}_P * ((\underline{r}_M - r_f) / \text{sigma}_M)$

where sigma_P (resp. sigma_M) is the standard deviation of the portfolio P (resp. the market portfolio.

So, we get $\text{sigma}_P = 17.6\%$

Answer 5:

Let the expected return on S_A be E_A and the variance of return be V_A. Then the expected return on S_B is $2E_A$ and the variance of return is $2V_A$.

(i) (a) The only zero risk portfolio can occur if the correlation is either 1 or -1. By considering diversification, the most efficient portfolio will occur when it is -1.

The overall portfolio variance is:

$$V = x_A^2 V_A + 2x_B^2 V_A + 2\sqrt{2}x_A x_B \rho V_A = V_A(x_A - \sqrt{2}x_B)^2$$
$$+2\sqrt{2}x_A x_B(1+\rho)V_A$$

Since $-1 \le \rho \le 1$ and $V_A > 0$
this can only be 0 if $\rho = -1$.

(b) Then, $V = 0 \Rightarrow x_A = \sqrt{2}x_B$ and the overall portfolio constrain $x_A + x_B = 1$ yields $x_A = \dfrac{\sqrt{2}}{\sqrt{2}+1}$ and $x_B = \dfrac{1}{\sqrt{2}+1}$.

(c) So the expected return on the overall portfolio $E = E_A.\dfrac{\sqrt{2}+2}{\sqrt{2}+1}$.

(ii) (a) In this case the maximum expected return is infinite (obtained by selling unlimited amounts of security S_A to purchase unlimited amounts of security S_B).

(b) In this case the maximum expected return is obtained by selling one unit of S_A to purchase two units of S_B. The maximum expected return is then $3\,E_A$.

(iii) In this case we have, using results from the core reading,

$$x_A = \frac{2V_A - 0.6V_A}{3V_A - 1.2V_A} = \frac{7}{9} = 0.7777 \text{ and so } x_B = \frac{2}{9} = 0.2222$$

And so the expected return is $\dfrac{11}{9}E_A = 1.2222\,E_A$.

Answer 6:

(i) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is $0.1A + 0.2B + 0.4C + 0.3D$ (asset E is the risk-free asset).

Asset	*A*	*B*	*C*	*D*	*E*	*Probability of being in state*
Annual return in						
State 1	3%	3%	3%	3%	3%	0.25
State 2	5%	7%	2%	8%	3%	0.5
State 3	7%	5%	8%	1%	3%	0.25
Market Capitalisation	10m	20m	40m	30m		

$E_A = 5\%$; $E_B = 5.5\%$; $E_C = 3.75\%$; $E_D = 5\%$

and so $E_M = (10\times5\%+20\times5.5\%+40\times3.75\%+30\times5\%)/100 = 4.6\%$

Now $\sigma^2_M = 0.25\times(3-4.6)^2+0.5\times(5.1-4.6)^2+0.25\times(5.2-4.6)^2 = 0.855\%$
and $\sigma_M = 0.92466\%$

(ii) market price of risk is $(E_M - r)/\sigma_M = (4.6 - 3)/0.92466 = 173\%$

(iii) $\beta_i = \text{Cov}(R_i, R_M)/\text{Var}(R_M)$.

Now $\text{Cov}(R_A, R_M) = 0.25\times3\times3+0.5\times5\times5.1+0.25\times7\times5.2-5\times4.6 = 1.1\%$;
$\text{Cov}(R_B, R_M) = 0.25\times3\times3+0.5\times7\times5.1+0.25\times5\times5.2-5.5\times4.6 = 1.3\%$;
$\text{Cov}(R_C, R_M) = 0.25\times3\times3+0.5\times2\times5.1+0.25\times8\times5.2-3.75\times4.6 = 0.5\%$;
$\text{Cov}(R_D, R_M) = 0.25\times3\times3+0.5\times8\times5.1+0.25\times1\times5.2-5\times4.6 = 0.95\%$

It follows that $\beta_A = 1.1/0.855=1.2865$, $\beta_B = 1.3/0.855 = 1.5205$,
$\beta_C = 0.5/0.855 = 0.5848$ and $\beta_D = 0.95/0.855 = 1.1111$.
OR
Assets all lie on the securities market line, so $E_i - r = \beta_i(E_M - r)$, so
$\beta_A = 2/1.6=1.25$, $\beta_B = 2.5/1.6 = 1.5625$
$\beta_C = 0.75/1.6 = 0.46875$ and $\beta_D = 2/1.6 = 1.25$.

(iv) Most of the assumptions of the basic model can be attacked as *unrealistic*. *Empirical studies do not provide strong support for the model.* There are basic *problems in testing the model* since, in theory, *account has to be taken of the entire investment universe open to investors,* not just capital markets.

Answer 7:

(i) The single-index model expresses the return on a security as:

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

where: R_i is the return on security i
α_i and β_i are constants
R_M is the return on the market

The ε_i are independent, zero-mean random variables, uncorrelated with R_M, representing the component of R_i not related to the market.

(ii) The expected return on security i is

$$E_i = \mathbb{E}(R_i) = \mathbb{E}(\alpha_i + \beta_i R_M + \varepsilon_i) = \alpha_i + \beta_i . E_M,$$

where E_M is the expected return on the market.

The variance of returns on security i is $V_i = Var(\alpha_i + \beta_i R_M + \varepsilon_i) = \beta_i^2 V_M + V_{\varepsilon_i}$, where V_M is the variance of returns on the market, V_{ε_i} is the variance of the random variable component of R_i not related to the market and the result holds because under the model ε_i is uncorrelated with R_M.

The covariance of returns between security i and security j is given by $C_{i,j} = Cov(R_i, R_j) = Cov(\alpha_i + \beta_i R_M + \varepsilon_i, \alpha_j + \beta_j R_M + \varepsilon_j) = \beta_i . \beta_j . V_M$, since under the model ε_i is uncorrelated with R_M and ε_i is independent of ε_j for all $i \neq j$.

(iii) Using the results from (ii), the variance of portfolio returns on a portfolio of N equally weighted securities is

$$V = \sum_{i,j=1}^{N} \frac{1}{N^2} Cov(R_i, R_j)$$

$$= \frac{1}{N^2} \sum_{i=1}^{N} (\beta_i^2 V_M + V_{\varepsilon_i}) + \frac{1}{N^2} \sum_{i,j=1, i \neq j}^{N} \beta i . \beta j . VM$$

$$= \left(\frac{1}{N} \sum_{i=1}^{N} \beta_i \right)^2 V_M \text{ plus terms which tend to zero as } N \to \infty.$$

In other words, the limiting portfolio variance depends on the average value of the β_i s and the variance of the market but not the specific risk of any individual security.

Alternative solution:

The single index model for a portfolio P of N assets held in proportions x_i, …, x_N is:

$$R_P = \alpha_P + \beta_P R_M + \varepsilon_P$$

where $\alpha_P = \sum_{i=1}^{N} x_i \alpha_i$, $\beta_P = \sum_{i=1}^{N} x_i \beta_i$ and $\varepsilon_P = \sum_{i=1}^{N} x_i \varepsilon_i$

So that (by the result in part (ii)):

$$V_P = \beta_P^2 V_M + V_{\varepsilon_P}$$

$$= \left(\sum_{i=1}^{N} x_i \beta_i\right)^2 V_M + \mathrm{var}\left(\sum_{i=1}^{N} x_i \varepsilon_i\right)$$

If $x_i = \dfrac{1}{N}$ then:

$$V_P = \frac{1}{N^2}\left(\sum_{i=1}^{N} \beta_i\right)^2 V_M + \frac{1}{N^2}\left(\sum_{i=1}^{N} V_{\varepsilon_i}\right)$$

$$= \bar{\beta}^2 V_M + \frac{1}{N}\bar{V}$$

Where $\bar{\beta}$ is the average of the individual β_i's and $\bar{V}$ is the average of the V_{ε_i}'s.

As $N \to \infty$, the second component, which represents the specific risk, tends to 0.

(iv) More factors will always improve the fit of a regression to historic data, in other words reduce the residual errors in relation to the data fitted, although market correlation typically has the most explanatory power.

There is little evidence that multi-factor models are significantly better at forecasting the future correlation structure.

Answer 8:

(i) The single-index model expresses the return on a security as

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

where R_i is the return on security i,

α_i and β_i are constants,

R_M is the return on the market,

ε_i is a random variable representing the component of R_i not related to the market.

(ii) The single-index model is purely empirical and is not based on any theoretical relationships between β_i and the other variables, which are assumed in CAPM.

(iii) The β_i are the ratio of the covariances of the securities with the market divided by the variance of the market.

So, $\beta_1 = \dfrac{15\times0.25\times10}{100} = 0.375$ and $\beta_2 = \dfrac{20\times0.4\times10}{100} = 0.8$.

Hence, $\alpha_1 = 8 - 0.375\times6 = 5.75$ and $\alpha_2 = 10 - 0.8\times6 = 5.2$.

(iv) As you are fitting more parameters, in-sample results should give a better fit (although not necessary a higher information criterion).

In terms of prediction, adding additional indices are unlikely to improve predictions, assuming the market is reasonably efficient.

Answer 9:

(i) The capital market line is given by

$$r_P - r_0 = \sigma_P / \sigma_M(r_M - r_0),$$

where

r_P is the expected return on an efficient portfolio, P;
r_M is the expected return on the market portfolio;
r_0 is the risk-free rate;
σ_P is the standard deviation of the return on the portfolio, P;
σ_M is the standard deviation of the return on the market portfolio.

(ii) r_P is 18% and so

$$14 = 8\sigma_P / 2, \text{ thus } \sigma_P = 0.035 = 3.5\%.$$

(iii) The efficient portfolio is a mix of the market portfolio and the risk-free asset. If the weights (which sum to 1) are w_M and w_0 then the expected return is $12w_M + 4\,w_0$ so $8\,w_M = 14$ and $w_M = 1.75$, $w_M = -0.75$.

Thus the efficient portfolio has £2,100,000 in the market portfolio and is short £900,000 in cash.

Answer 10:

A strand of empirical research questions the use of the normality assumptions in market returns. In particular,

- market crashes appear more often than one would expect from a normal distribution. While the random walk produces continuous price paths, jumps or discontinuities seem to be an important feature of real markets.

- Furthermore, days with no change, or very small change, also happen more often than the normal distribution suggests. This would seem to justify the consideration of Levy processes.

- Q-Q plots of the observed changes in the FTSE All Share index against those which would be expected if the returns were lognormally distributed show substantial differences. This demonstrates that the actual returns have many more extreme events, both on the upside and downside, than is consistent with the lognormal model.

- a quintic polynomial distribution whose parameters have been chosen to give the best fit to the data, clearly provides an improved description of the returns observed, in particular more extreme events are observed than is the case with the lognormal model. The rolling volatilities of a simulation from the non-normal distribution show significant differences over different periods. This volatility process has the same characteristics as the observed volatility from the equity market.

Answer 11:

(i) S has a higher return and a lower variance so is preferable in a mean-variance framework. [2]

(ii) You can relax the assumption that investors solely select their portfolios on the basis of the expected return and variance of that return. [1]

(iii) $E[P] = 0.75 \times E[S] + 0.25 \times E[M] = 9.5\%$

$\text{Var}(P) = 0.75^2\ \text{Var}(S) + 0.25^2\ \text{Var}(M) + 2 \times 0.75 \times 0.25 \times 4 \times 5 \times 0.3$
$= 12.8125\%\%$

So standard deviation $(P) = \sqrt{(12.8125)}\ = 3.57945\%$ [2]

(iv) The amount invested in S, x_S, will be,

$$x_S = \frac{V_M - C_{SM}}{V_S - 2C_{SM} + V_M} = \frac{25 - 4\times5\times0.3}{16 - 2\times4\times5\times0.3 + 25} = 0.655173$$

invested in S, and so 0.344828 invested in M. [2]

(v) The study suggests that the correlation between M and S will increase.

This means that portfolios containing positive amounts of M and S will have a higher variance.

If the correlation increases, then the minimum variance portfolio will contain relatively higher amounts of S and relatively lower amounts of M. [3]

Answer 12:

(i) Investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon.

The expected returns, variance of returns and covariance of returns are known for all assets and pairs of assets.

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return.

(ii) Let the proportion invested in asset i, be x_i, with expected return E_i, variance V_i and correlation ρ12. Let E be the return on the portfolio of the three assets and let λ and μ be Lagrange multipliers.

Then, the Lagrangian function W satisfies:

$$W = \sum_{i=1}^{3} x_i^2 V_i + 2\rho_{13}\sigma_1\sigma_3 x_1 x_3 - \lambda(E_1x_1 + E_2x_2 + E_3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

$$= 16x_1^2 + 144x_2^2 + 64x_3^2 + 48x_1x_3 - \lambda(2x_1 + 4x_2 + 3x_3 - E) - \mu(x_1 + x_2 + x_3 - 1)$$

(iii) $$\frac{\partial W}{\partial x_1} = 32x_1 + 48x_3 - 2\lambda - \mu = 0 \quad \frac{\partial W}{\partial x_2} = 288x_2 - 4\lambda - \mu = 0$$

$$\frac{\partial W}{\partial x_3} = 128x_3 + 48x_1 - 3\lambda - \mu = 0$$

Substituting the values given for x_i, we obtain three equations for λ and μ, solving these gives $\lambda = 64.8$ and $\mu = -100.8$ and we can check that these values satisfy the constraints.

(iv) Without short selling, the only way to get an expected return of 4% is to invest wholly in asset 2.

Answer 13:

(i) A portfolio is efficient if the investor cannot find a better one in the sense that it has a higher expected return with the same variance, or a lower variance with the same expected return.

(ii) The assumptions are:

Investors are never satiated.

Investors dislike risk.

Investors select assets based on mean and variance of return only.

Mean return, variance (or standard deviation) and co-variances are known for all assets.

(iii) $E = \sum_i x_i E_i$ where E_i is the expected return on security i.

(iv) $V = \sum_i \sum_j x_i x_j C_{ij}$ where C_{ij} is the covariance of the returns on securities i and j and we write $C_{ii} = V_i$.

(v) With only two securities the variance is

$$V = x_A^2 V_A + (1-x_A)^2 V_B + 2x_A(1-x_A)C_{AB}$$

Differentiating wrt x_A gives

$$\frac{dV}{dx_A} = 2x_A V_A - 2(V_B - x_A V_B) + 2(1-2x_A)C_{AB}$$

$$= (2V_A + 2V_B - 4C_{AB})x_A - 2V_B + 2C_{AB}$$

Setting this to zero gives

$$x_A = \frac{V_B - C_{AB}}{V_A + V_B - 2C_{AB}}$$

Checking the second derivative shows that this is a minimum:

$$\frac{d^2V}{dx_A^2} = 2V_A + 2V_B - 4C_{AB} \geq 0$$

Answer 14:

(i) The composition of the market portfolio is as follows:

Market capitalisation	30,000	70,000
w_i	0.3	0.7

(ii) Mean returns: Asset 2: 21.8% Asset 3: 14.9%

Consequently:

$$Er_M = \sum_{i=2}^{3} w_i Er_i = 16.97\%$$

$$\text{std. dev}(r_M) = \sigma_M = \sqrt{E\left[\left(\sum_{i=2}^{3} w_i r_i\right)^2\right] - (Er_M)^2} = 3.57\%.$$

The market price of risk is given by $(Er_M - r_f)/\sigma_M$

And since the risk-free rate is 5.0%, this equates to:

(0.1697 – 0.05)/0.0357 = 3.35

(iii) From the Security Market Line it follows that $\beta_i = (Er_i - r_f)/(Er_M - r_f)$. [1]

Hence $\beta_2 = 1.40$ and $\beta_3 = 0.83$. [1 mark each]
[Max 2]

(iv) The assumptions made are unrealistic. [1]
Empirical studies do not provide strong support for the model. [1]
It does not account for taxes. [1]
Or inflation. [1]
Or situations in which there is no riskless asset. [1]
It does not consider multiple time periods. [1]
Or optimisation of consumption over time. [1]
Investors don't always use the same "currency" [1]
Markets are not always perfect [1]
Investors don't always have the same expectations [1]
Cannot lend/borrow unlimited amounts at the same risk-free rate [1]
Difficult to check as need to think about investment markets as well as capital markets [1]
Unrealistic to invest in the market portfolio in practice as so many stocks [1]

Answer 15:

(i) Investors select their portfolios on the basis of the expected return and the variance of that return over a single time horizon. [1]

The expected returns, variance of returns and covariance of returns are known for all assets and pairs of assets. [1]

Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return. [1]

Investors dislike risk. For a given level of return they will always prefer a portfolio with lower variance to one with higher variance. [1]

(ii) We use the following notation for i=A,B:

$E(S_i)=E_i$
$V(S_i)=V_i$

and C_{AB} is the covariance between the returns of Asset A and Asset B.

(a) From the Core Reading

$$x_A = \frac{V_B - C_{AB}}{V_A - 2C_{AB} + V_B}.$$

So $x_A = (0.25V_A - 0) / (V_A - 0 + 0.25V_A) = 0.2$

and $x_B = 0.8$.

Hence expected return $= 0.2 \times E_A + 0.8 \times 0.25E_A$

$= 0.4E_A$.

(b) Now $C_{AB} = \sqrt{(V_A \times 0.25 \times V_A)} = 0.5V_A$.

So $x_A = (0.25V_A - 0.5V_A) / (V_A - 2 \times 0.5V_A + 0.25V_A) = -1$

and $x_B = 2$.

Hence expected return $= -1 \times E_A + 2 \times 0.25E_A$

$= -0.5E_A$.

(iii) (a) The variance of the return on the portfolio in (b) is:

$(-1)^2 \times V_A + 2^2 \times V_B + 2 \times (-1) \times 2 \times 0.5V_A$

$= 0$.

(b) So we have created a risk-free portfolio.

Answer 16:

(i) The market portfolio is the weighted portfolio of the risky securities in the market,
consequently $Er_M = 18\% = w_1 Er_1 + w_2 Er_2$.

As $w_1 + w_2 = 1$, then $w_1 = w_2 = 0.5$.

(ii) From the Security Market Line:

$$\beta_i = \frac{Er_i - r_f}{Er_M - r_f}$$

therefore $\beta_A = 1.25$ and $\beta_B = 0.75$.

Answer 17:

(i) $R_i = a_i + b_{i,1}\ I_1 + b_{i,2} I_2\ + c_i$

where a_i and c_i are the constant and random parts respectively of the component of the return unique to security *i*

I_1, I_2 are the changes in a set of the two indices

$b_{i,k}$ is the sensitivity (factor beta) of security *i* to factor *k*

(ii) **Macroeconomic factor models**

These use observable economic time series as the factors, such as the annual rates of inflation and economic growth, short term interest rates, the yields on long term government bonds, and the yield margin on corporate bonds over government bonds. [1]

Fundamental factor models

These use company specific variables as the factors, e.g. the level of gearing, the price earnings ratio, the level of research and development spending, the industry group to which the company belongs. [1]

Statistical factor models

These do not rely on specifying the factors independently of the historical returns data. Instead a technique called principal components analysis can be used to determine a set of indices which explain as much as possible of the observed variance. [1]

Answer 18:

(i) The market portfolio is the weighted portfolio of the risky securities in the market, consequently

$$Er_M = 18\% = w_A Er_A + w_B Er_B$$

As $w_A + w_B = 0.5$, then $w_A = w_B = 0.5$.

(ii) From the security market line

$$\beta_i = \frac{Er_i - r_f}{Er_M - r_f}$$

Therefore $\beta_A = 1.2$ and $\beta_B = 0.8$. [1 ea

(iii) Empirical studies do not provide strong support for the model. [

The underlying assumptions are not realistic. [

Investors cannot necessarily borrow or lend unlimited amounts at the same risk-free rate. [

The markets for risk assets may not be perfect. [

Investors may not have the same estimates of expected returns, standard deviations and covariances of securities. There are basic problems in testing the model since, in theory, account has to be taken of the entire investment universe open to investors, not just capital markets. [

It does not account for taxes. [

It does not account for inflation. [Or: some investors may measure in real terms and some in money terms.] [!

It does not account for situations in which there is no riskless asset. [!

The basic model does not allow for currency risk. [Or: investors may not measure in the same currency.] [!

It does not consider multiple time periods. [Or: investors do not all have the same one-period time horizon.] [!

It does not consider optimisation of consumption over time. [!

Answer 19:

(i) (a) A portfolio is inefficient if the investor can find another portfolio with the same expected return and lower variance,

or the same variance and higher expected return.

(b) A portfolio is efficient if the investor cannot find a better one in the sense that it has both the same or higher expected return

and the same or lower variance.

(ii) The assumptions are:

(a) Investors are never satiated. [At a given level of risk, they will always prefer a portfolio with a higher expected return to one with a lower return.]

(b) Investors dislike risk. [For a given level of return, they will always prefer a portfolio with lower expected variance to one with higher variance.]

(iii) $V = a^2V_A + (1-a)^2V_B + 2a(1-a)(V_AV_B)^{0.5}C_{AB}$

$= 0.16a^2 + 0.25\,(1-a)^2 - 2a(1-a)\,(0.16 \times 0.25)^{0.5} \times 0.2$

$= 0.49a^2 - 0.58a + 0.25$

(iv) $R = aR_A + (1-a)R_B$
$= -0.02a + 0.07$

So $R^2 = 0.0004a^2 - 0.0028a + 0.0049$

So $V = 1225R^2 - 142.5R + 4.2225$

(v) $1225R^2 - 142.5R + 4.2225 = 16R - 200R^2$

So $R = 0.0670$ or 0.0442

Hence $a = 0.1497$ or 1.2889

(vi) The second solution implies a proportion of –0.2889 invested in asset B

so would not be allowed, hence only the first solution would remain.

Answer 20:

i) CAPM assumptions [½ mark each]

a. All investors have the same one-period horizon.
b. All investors can borrow or lend unlimited amounts at the same risk-free rate.
c. The markets for risky assets are perfect. Information is freely and instantly available to all investors and no investor believes that they can affect the price of a security by their own actions.
d. Investors have the same estimates of the expected returns, standard deviations and covariances of securities over the one-period horizon.
e. All investors measure in the same "currency" e.g. pounds or dollars or in "real" or "money" terms.

ii) By definition the beta of each security is $\beta_i = Cov(R_i, R_M)/Var(R_M)$ [½
where R_i is the rate of return on security i, $R_M, Var(R_M)$ are respectively the rate of return on the market portfolio and its variance [½
Hence

$$\beta_M = \frac{Cov(R_M, R_M)}{Var(R_M)} = 1$$

as required [1
(Note to markers: the same conclusion can be reached from the Security Market Line, and is equally acceptable)

iii) As the market portfolio is the weighted portfolio of the risky securities in the market, and the given weights are 0.4 and 0.6, then

$$Cov(R_i, R_M) = Cov(R_i, 0.4R_1 + 0.6R_2)$$

from which it follows that [½ marl

$$Cov(R_1, R_M) = 0.4Var(R_1) + 0.6Cov(R_1, R_2) = 0.00089$$
$$Cov(R_2, R_M) = 0.4Cov(R_1, R_2) + 0.6Var(R_2) = 0.00150$$

Also:

$$Var(R_M) = 0.4^2 * Var(R_1) + 0.6^2 * Var(R_2) + 2 * 0.4 * 0.6 * Cov(R_1, R_2) = 0.00125$$

Consequently $\beta_1 = 0.70915, \beta_2 = 1.1939$ [½ each]

[Note to Markers: please accept any correct attempt with rounded figures. For

$$Cov(R_1, R_M) = 0.4Var(R_1) + 0.6Cov(R_1, R_2) = 0.0009$$

we obtain $\beta_1 = 0.72, \beta_2 = 1.2$*]*

iv) From the Security Market Line it follows that $R_f = (ER_i - \beta_i ER_M)/(1 - \beta_i)$
From the data $ER_1 = 6.40\%, ER_2 = 9.90\%$. [½ marks each]

Consequently
$ER_M = \sum_{i=1}^{2} w_i ER_i = 8.5\%$ [½]

and $R_f = 0.012797$ [½]

[Note to Markers: if the rounding above and the corresponding betas are used, then from the equation for asset 1 we obtain $R_f = 0.01$*, whilst from the equation for asset 2 we obtain* $R_f = 0.015$*. Please accept any valid attempt.]*

Answer 21:

i. SML: $ER_i = R_f + \beta_i(ER_M - R_f)$
for

- ER_i: expected return on Asset i.
- R_f: risk-free rate.
- β_i: beta factor of security i defined as $Cov(R_i, R_M)/Var(R_M)$.

- ER_M: expected return on the market portfolio. [1/4]
  [Round up to nearest half mark.]

ii. Note that $ER_M = x_1 ER_1 + x_2 ER_2$ [1/4], and $x_1 + x_2 = 1$ [1/4]
Substitute into the SML and solve for x_1, so that
$x_1 = \frac{ER_i - \beta_i ER_2 - R_f(1-\beta_i)}{(ER_1 - ER_2)\beta_i}$. [1]
From the data: $ER_1 = 16.30\%$ [1/4]
$ER_2 = 29.70\%$ [1/4]
Substituting either for Asset 1 or Asset 2, $x_1 = 0.4$ [1/2]
and therefore $x_2 = 0.6$ [1/2]
*[Alternatively, the beta of the market portfolio is 1, so x_1*0.46 + x_2*1.36 = x_1*0.46 + (1-x_1)*1.36 = 1 => x_1=0.4]*
[Round up to nearest half mark.]

iii. $Var(R_M) = 0.4^2 * Var(R_1) + 0.6^2 * Var(R_2) + 2 * 0.4 * 0.6 * Cov(R_1, R_2) = 0.00617.$ [1]
Consequently $\left(ER_M - R_f\right)/\sigma_M = 1.897.$ [1]

Answer 22:

(i) A portfolio is efficient if the investor cannot find a better one [½]
…in the sense that it has a higher expected return for the same variance or a lower variance for the same expected return. [½]

(ii) Investors are never satiated. At a given level of risk, they will always prefer a portfolio with a higher return to one with a lower return. [1]

Investors dislike risk. For a given level of return, they will always prefer a portfolio with lower variance to one with higher variance. [1]

(iii) The portfolio is inefficient [0.5]
So we can find another portfolio with the same expected return but lower risk [0.5]
Or the same risk but a higher expected return [0.5]
[Max 1]

(iv) Variance = $V = x_A^2 V_A + x_B^2 V_B + 2x_A x_B C_{AB} = x_A^2 V_A + (1-x_A)^2 V_B + 2x_A(1-x_A)C_{AB}$
$= (V_A+V_B-2C_{AB})x_A^2 + (2C_{AB}-2V_B)x_A + V_B$ [1]
$dV/dx_A = 2(V_A+V_B-2C_{AB})x_A + 2C_{AB}-2V_B$ [1]
$dV/dx_A = 0$ iff $x_A = (V_B-C_{AB}) / (V_A+V_B-2C_{AB})$ [1]
$d^2V/dx_A^2 = 2(V_A+V_B-2C_{AB}) > 0$ because $C_{AB} <= (V_A+V_B)/2$ hence this is a minimum [1]

Answer 23:

(i)

(a)

$\beta_i = \text{Cov}(R_i,R_M)/\text{Var}(R_M)$.

(b)

$r_i - r_0 = \beta i\ (r_M - r_0)$

where r_0 is the return on the risk-free asset.

(c)

Since $R_M = \Sigma\pi_i R_i$, it follows that

$\text{Var}(R_M) = \Sigma\pi_i \text{Cov}(R_i, R_M)$

and so $\Sigma\pi_i \beta_i = \Sigma\pi_i \text{Cov}(R_i, R_M)/\text{Var}(R_M)$

$= \text{Var}(R_M)/\text{Var}(R_M) = 1$

(ii)

(a)

The proportions are given by proportions of market capitalisation so that

$\pi_1 = 2/5$, $\pi_2 = \pi_3 = \pi_4 = 1/5$.

(b)

$\text{Cov}(R_1, R_M) = 2/5 \times 4 + 1/5 \times 1 + 1/5 \times 1 + 1/5 \times 1 = 11/5$, so $\beta_1 = (11/5) / (8/5) = 11/8$.

Similarly, $\text{Cov}(R_2, R_M) = 2/5 \times 1 + 1/5 \times 3 + 1/5 \times 1 + 1/5 \times 1 = 7/5$, so $\beta_2 = 7/8$

and $\text{Cov}(R_3, R_M) = 2/5 \times 1 + 1/5 \times 1 + 1/5 \times 2 + 1/5 \times 1 = 6/5$, so $\beta_3 = 6/8$.

[½]

Now it follows from (i)(c) (i.e. $\Sigma\pi_i\beta_i$=1) that $\beta_4 = 5/8$.

(c)

We conclude that, since $r_i - r_0 = \beta_i\ (r_M - r_0)$,

$11\% = 11/8 \times (r_M - r_0)$ [0.5] so that $r_M = 11\%$

Then:

Asset number	1	2	3	4	Market
Expected return	14%	**10%**	**9%**	**8%**	11%

Answer 24:

(i)

$MPR = \frac{expected\ return - 5\%}{standard\ deviation}$

(ii)

The expected return on the portfolio is:

$$E = x_A E_A + x_B E_B$$

Substituting $xA=1-xB$ into this gives:

$$E = 0.1(1 - x_B) + 0.05x_B = 0.1 - 0.05x_B$$

Rearranging this gives:

$$x_B = \frac{0.1-E}{0.05} \text{ and } x_A = 1 - x_B = \frac{E-0.05}{0.05}$$

We also have:

$$\sigma^2 = x_A{}^2 V_A + x_B{}^2 V_B + 2x_A x_B C_{AB}$$

$$= 0.2^2 x_A{}^2$$

i.e. $\sigma = 0.2x_A$

Substituting our formula for xA into this gives the equation of the efficient frontier as:

$$\sigma = 0.2\left(\frac{E-0.05}{0.05}\right) = 4(E - 0.05)$$

This is a straight line in $E-\sigma$ space

Furthermore:

when $E=0.05, \sigma=0$ and

when $E=0.075, \sigma=0.1$ as required

(iii)

Asset B is risk-free, so it has zero standard deviation, and any combination of Asset B and a portfolio of assets involving A and C only will lie somewhere along the straight line in the $E-\sigma$ space joining Asset B and the other portfolio. [1]

The efficient frontier involving all three assets will be the one that maximises the market price of the risk (MPR). This must be the straight-line through Asset B that is tangential to the efficient frontier involving Assets A and C only. [1]

At the point of tangency:

- we have no holding of Asset B, $x_B = 0$ [½]

- $x_A = \frac{5}{9}$ *and* $x_C = \frac{4}{9}$ [

Here:

$$E = 0.1 \times \frac{5}{9} + 0.06 \times \frac{4}{9} = \frac{37}{450}$$

[

$$\sigma = \sqrt{0.04\left(\frac{5}{9}\right)^2 + 0.01\left(\frac{4}{9}\right)^2} = \frac{\sqrt{116}}{90}$$

[

So, the equation of the efficient frontier involving all three assets is:

$$E = 0.05 + \left(\frac{\frac{37}{450} - \frac{1}{20}}{\frac{\sqrt{116}}{90}}\right)\sigma = 0.05\left(1 + \sigma\sqrt{29}\right)$$

|

When σ=0.1:

$$E = 0.05\left(1 + 0.1 \times \sqrt{29}\right) = \frac{1}{20} + \frac{\sqrt{29}}{200} = 0.07693$$

[

So the efficient frontier involving all three assets passes through the point (0.1, 0.0769) as required. [

Answer 25:

(i)
(10/80)*5% + (50/80)*10% + (20/80)*x = R_M
x – 3% = 2(RM – 3%) => x = 2*RM – 3%
=> 10*5% + 50*10% + 20*(2*RM – 3%) = 80* R_M
=> R_M = (10*5% + 50*10% - 20*3%) / 40 = 12.25%

OR
(10% - 3%) / (E_M – 3%) = 1
=> R_M = 10%

Or full credit for any other valid approach.

(ii)
x – 3% = 2*(12.25% - 3%)
=> x = 21.5%

OR
x – 3% = 2*(10% - 3%)
=> x = 17%

(iii)
$5\% - 3\% = \beta*(12.25\% - 3\%)$
$=> \beta = 0.216$

OR
$5\% – 3\% = \beta *(10\% - 3\%)$
$=> \beta = 0.286$

Or full credit for any other valid approach.

(iv) There are basic problems in testing the model since, in theory, account has to be taken of the entire investment universe open to investors, not just capital markets

An important asset of most investors, for example, is their human capital (i.e. the value of their future earnings). Models have been developed which allow for decisions over multiple periods and for the optimisation of consumption over time to take account of this

Other versions of the basic CAPM have been produced which allow for taxes, inflation, and also for a situation where there is no riskless asset

In the international situation there is no asset which is riskless for all investors (due to currency risks) so a model has been developed which allows for groups of investors in different countries, each of which considers their domestic currency to be risk-free

There is a discrepancy between the values obtained for the expected return on the market using the security market line and the weightings by market capitalisation

Some other assumptions of CAPM are unrealistic, e.g. everyone has the same estimates for the means/variances/covariances, everyone has the same single time horizon

Answer 26:

(i)
We know that $E_i = r + \beta_i * (E_M - r)$
So $3.5 = r + 1.5 * (E_M - r)$

And $2.2 = r + 0.2 * (E_M - r)$

Subtracting the second equation from the first:
$3.5 - 2.2 = 1.3 * (E_M - r)$
So $E_M - r = 1$

Also subtracting 7.5 times the second equation from the first:
$3.5 - 7.5 * 2.2 = -6.5r$
So $r = 2\%$
And $E_M = 3\%$

(ii)
$MPR = (E_M - r) / \sigma_M = (3\% - 2\%) / 10\% = 0.1$

(iii)
Every asset must have an expected return at least as high as the risk-free rate
So the return on asset B must increase to at least 3%
Intuitively if the risk-free rate increases, then investors will require a higher return on risky assets as well
If the market price of risk remains unchanged, then the returns on all assets might just increase by 1%