Lecture

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: SY BSc

Subject : Portfolio Theory and Security Analysis

Subject Code:

Chapter: Unit 1 Chapter 2

Chapter Name: Economic Approach to Human Behavior

Today's Agenda

1. Saint Petersburg Paradox
2. Define Utility
3. Expected Utility theorem
   1. Four Axioms for Expected Utility Theorem
4. Non-satiation
5. Fair gamble
6. Types of Investors
7. Risk aversion and certainty equivalent
8. Types of gamble
9. Measuring risk aversion
10. Utility function
11. Iso elastic utility functions
12. Limitation of Utility theory
13. Construction of Utility function
14. Maximizing Utility Through Insurance
15. Finding The Minimum Premium

1 Saint Petersburg Paradox

In Saint Petersburg game, a coin is tossed; if "head" comes up the first time, the player receives two dollars($) and zero otherwise; if "head" appears also the second time, the player receives four dollars($) and zero otherwise; the third time, eight $, etc.

How much should a player pay to be able to participate in this game?

2 Define Utility

- *Utility*: is the satisfaction that an individual obtains from a particular course of action or it can refer to the total satisfaction received from consuming a good or service.
- In simple words, Utility is happiness achieved from having a certain amount of money.

In the application of utility theory to finance and investment choice, it is assumed that a numerical value called the utility can be assigned to each possible value of the investor's wealth by what is known as a preference function or utility function.

3 Expected Utility Theorem

A function U(w) can be constructed to represent an **investor's utility of wealth**, w, at some future date and the investor will make decisions on the basis of **maximizing expected utility.**

3.1 Four Axioms for Expected Utility Theorem

1. Comparability:

An investor can state a preference between all available certain outcomes. In other words, for any two certain outcomes A and B, either:

1. A is preferred to B
2. B is preferred to A
3. The investor is indifferent between A and B.

These preferences are sometimes denoted by:

1. $U(A) > U(B)$,
2. $U(B) > U(A)$
3. $U(A) = U(B)$

A & B are examples of levels of wealth.

3.1 Four Axioms for Expected Utility Theorem

2. Transitivity:

If A is preferred to B and B is preferred to C, then A is preferred to C.
i.e. if $U(A) > U(B)$ and $U(B) > U(C)$ then $U(A) > U(C)$

3. Independence:

If an investor is indifferent between two certain outcomes, A and B, then they are also indifferent between the following two gambles:

- A with probability p and C with probability (1 - p)
- B with probability p and C with probability (1 - p)

.

3.1 Four Axioms for Expected Utility Theorem

4. Certainty equivalence

- Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability (1 - p)

Thus if:

$$U(A) > U(B) > U(C)$$

Then there exists a unique p ($0 < p < 1$) such that :

$$pU(A) + (1 - p)U(C) = U(B)$$

B is known as the 'certainty equivalent' of the above gamble

4 Non-satiation (Never satisfied)

The principle of Non Satiation is the assumption that people prefer more wealth to less.

Mathematically, it can be expressed as: $\mathbf{U'(w) > 0}$

The derivative of utility with respect to wealth is often referred to as the *marginal utility of wealth*.

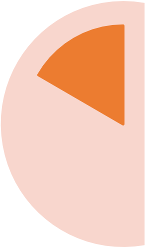
Non-satiation is therefore equivalent to an assumption that the marginal utility of wealth is strictly positive.

5 Fair Gamble

A *fair gamble* is one that leaves the expected wealth of the individual unchanged. Equivalently, it can be defined as a gamble that has an overall expected value of zero.

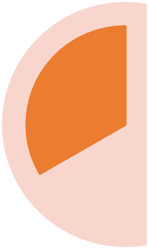
6 Types of Investors

Risk Averse



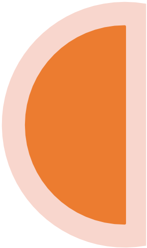
- Hates risk, hence rejects a fair gamble
- As the wealth increases, places less value on a fixed increase in wealth

Risk Neutral



- Neutral to risk, hence indifferent to a fair gamble
- As the wealth increases, the individual is indifferent on the fixed increase in wealth

Risk Seeker



- Loves risk, hence accepts a fair gamble
- As the wealth increases, places more value on a fixed increase in wealth

6 Types of Investors – Based on Utility Function

Risk Averse

- values an incremental increase in wealth less
- will reject a fair gamble

$U''(w) < 0$

Risk Neutral

- Is indifferent between a fair gamble and the *status quo*

$U''(w) = 0$

Risk Seeker

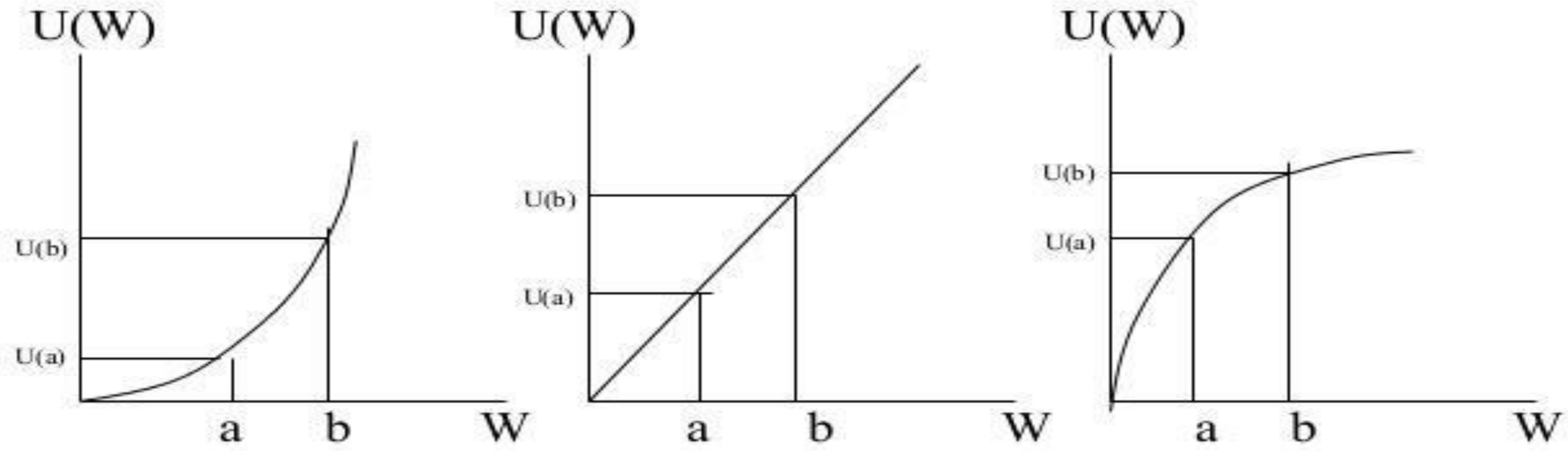
- values an incremental increase in wealth more
- will seek a fair

gamble

$U''(w) > 0$

6 Types of Investors - Based on Utility Function

Preferences to Risk



Risk Preferring

$U'(W) > 0$
$U''(W) > 0$

Risk Neutral

$U'(W) > 0$
$U''(W) = 0$

Risk Aversion

$U'(W) > 0$
$U''(W) < 0$

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7 Risk Aversion and the certainty equivalent

For a risk averse investor, the certainty equivalent of a fair gamble is negative, i.e. the investor would have to be paid to accept the gamble.

We can distinguish between two different types of certainty equivalents depending upon the situation that we are considering:

- The certainty equivalent of the portfolio consisting of the combination of the existing wealth *w and* the gamble *x*, which we can denote c_w
- The certainty equivalent of the gamble *x* alone, c_x

Thus, for a fair gamble and a risk-averse investor, it must be the case that

$$c_w < w \text{ and } c_x < 0$$

8 Types of Gamble

Additive Gamble

- sums won or lost are *fixed absolute amounts*.
- $U(c_w) = E[U(w + x)]$
- CE(Gamble) = CE(Wealth + Gamble) - Wealth

Multiplicative Gamble

- outcomes represented by a random variable *y*, in which the sums won or lost are all expressed as *proportions* of the initial wealth
- $U(c_w) = E[U(w \times y)]$
- CE(Gamble) = CE(Wealth + Gamble) - Wealth

Question

CT8, May 2010, Q.6

Ram's utility U(w) for a given level of wealth 'w' is determined by the following function:

$$U(w) = \begin{cases} 4\left(\dfrac{w}{10000}\right)^2, & w \le 5{,}000 \\[2ex] \dfrac{3w}{5000} - 2, & 5{,}000 < w \le 10{,}000 \\[2ex] \log_{10} w, & w > 10{,}000 \end{cases}$$

Question

(i) State the principle of non-satiation in the context of an investor.
(ii) Is Ram a non-satiated investor?
(iii) Define the following types of investors including the respective behavior of their utility function
• risk averse investor • risk seeking investor • risk neutral investor

Solution

Answer:

(i) It is usually assumed that people prefer more wealth to less. This is known as the principle of non-satiation and can be expressed as:

$U'(w) > 0$

(ii) Ram's utility function is given by

$$U(w) = \begin{cases} 4\left(\frac{w}{10000}\right)^2, & w < 5{,}000 \\ \frac{3w}{5000} - 2, & 5{,}000 < w \le 10{,}000 \\ \log_{10} w, & w > 10{,}000 \end{cases}$$

Solution

This implies

$$U'(w) = \begin{cases} \dfrac{8w}{10000^2}, & w \le 5{,}000 \\ \dfrac{3}{5000}, & 5{,}000 < w \le 10{,}000 \\ \dfrac{1}{w \log_e 10}, & w > 10{,}000 \end{cases}$$

Ram's marginal utility is strictly positive at all levels of wealth. We can therefore say that Ram is a non satiated investor.

Solution

(iii) Risk Averse Investor - A risk-averse investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. The utility function condition is:

$U''(w) < 0$

Risk Seeking Investor - A risk-seeking investor values an incremental increase in wealth more highly than an incremental decrease and will seek a fair gamble. The utility function condition is:

$U''(w) > 0$

Risk Neutral Investor - A risk-neutral investor is indifferent between a fair gamble and the status quo. In this case:

$U''(w) = 0$

9 Measuring Risk Aversion

Absolute Risk Aversion

$$A(w), ARA = -\frac{U''(w)}{U'(w)}$$

Relative Risk Aversion

$$R(w), RRA = -w\frac{U''(w)}{U'(w)}$$

These formulae are given in Tables

These are often referred to as the ***Arrow-Pratt*** measures of absolute risk aversion and relative risk aversion.

Question

CT8, May 2012 ,Q.2

Consider the following utility function:

$$U(w) = -\frac{1}{\sqrt{w}}$$

Where U: Utility w: wealth

a) Derive the expression for the absolute risk aversion and relative risk aversion measures.

b) Assuming that the wealth of the investor increases, interpret the values of the coefficients of absolute risk version and relative risk aversion (as computed in part a) in terms of investment in risky assets.

Question

c) Suppose that the investor (with an initial wealth of Rs. 6.5 million) is offered the following four investment portfolios with a payoff in year 1 as described below.

Investment A		Investment B		Investment C		Investment D	
Payoff	Probability	Payoff	Probability	Payoff	Probability	Payoff	Probability
5	1/3	4	1/4	1	1/5	1	4/10
6	1/3	7	1/2	9	3/5	6.5	5/10
7	1/3	10	1/4	18	1/5	31.8	1/10

Which is the most preferred investment portfolio (if any) of the investor given the above utility function?

Solution

Answer:

a. u(w) = 1/w^0.5
Differentiating
u'(w) = w^(-3/2)/2

u''(w) = (-3/4)w^(-5/2)

Co-efficients
A(w) = -U″(w) / U′(w)
R(w) = wA(w)

Therefore,
A(w) = (3/2w)
R(w) = 3/2

Solution

b. To study the effect

Calculate

$A'(w) = (-3/2w^2) < 0$

$R'(w) = 0 = 0$

This shows that investor exhibits decreasing ARA i.e
So as wealth increases he will hold more dollars in risky assets.
However constant RRA implies as a % invested in risky assets is unchanged as wealth increases.

Solution

c.

A			B		
Outcome	Probability	Utility	Outcome	Probability	Utility
5	0.33	-0.45	4	0.25	-0.50
6	0.33	-0.41	7	0.50	-0.38
7	0.33	-0.38	10	0.25	-0.32

C			D		
Outcome	Probability	Utility	Outcome	Probability	Utility
1	0.20	-1.00	1	0.40	-1.00
9	0.60	-0.33	6.5	0.50	-0.39
18	0.20	-0.24	31.8	0.10	-0.18

EU (A)	0.41114
EU (B)	0.39304
EU (C)	-0.44714
EU(D)	0

Solution

Where Expected U (i) = Σ pi* ui

The investor is risk averse. If, however, the investor does not invest in any of the portfolios, then his or her expected (and certain) utility is

$$-\frac{1}{\sqrt{6.5}} = -0.3922$$

Thus, as investing in any of the portfolio gives the investor a lower expected utility, he or she will not invest in any of the portfolios.

10 Utility Functions

1. Quadratic: $U(w) = w + dw^2$
2. Log: $U(w) = ln(w)$
3. Power: $U(w) = (w^{\gamma} - 1)/\gamma$

11 Iso Elastic Utility Functions

An iso elastic utility function is one that exhibits constant RRA, i.e. $R'(w) = 0$

Log and power utility functions are iso-elastic

Question

CT8, November 2011 ,Q.1

Assume an investor has initial wealth of 1 unit and his utility function has the following form:

$$U(w) = \frac{w^{\gamma} - 1}{\gamma} \text{ for } w > 0$$

i) For what values of γ will the investor be considered as risk averse?

ii) Explain what is meant by an iso-elastic utility function and prove that the investor's utility function is iso-elastic.

Solution

Answer:

The investor's utility is depicted through a power utility function.

i) For a risk averse investor, the utility function condition is

$$U''(w) < 0$$

For power utility function we have

$$U'(w) = w^{\gamma-1}, \text{and } U''(w) = (\gamma - 1)w^{\gamma-2}$$

Thus, to model the behaviour of a risk averse investor we should have $\{\gamma < 1;\ \gamma \neq 0\}$.

Solution

ii) Iso-elastic utility functions display constant elasticity of marginal utility (with respect to wealth) as wealth increases. Such utility functions exhibit constant relative risk aversion,

$$R(w) = \text{constant and } R'(w) = 0$$

where R(w) is defined as

$$R(w) = -w\frac{U''(w)}{U'(w)}$$

For power utility function

$$R(w) = (1-\gamma) \text{ and } R'(w) = 0$$

Hence the power utility function is iso-elastic.

Question

CT8, May 2011, Q.10

An individual who prefers more to less has a quadratic utility function and initial wealth of 100. She faces a random loss that is normally distributed with mean 5 and standard deviation 10. She is indifferent between facing this loss and paying 5.5 to fully protect herself from the loss.

a) Find the form of her utility function.

b) The individual is considering entering a lottery in which the first prize is 1000. Explain whether the utility function from (a) can be used to calculate how much should be paid for the ticket.

Solution

Solution:

(a)

Let x be the loss. Hence equivalence of

$$E[u(100 - x)] = u[94.5]$$

for $u(w) = w + dw^2$ we have

$$100 - E(x) + d(100^2 - 2E(x)100 + E(x^2)) = 94.5 + d94.5^2$$

$$d = (94.5 - 100 + E(x))/(100^2 - 2E(x)100 + E(x^2) - 94.5^2)$$

$$d = -0.002567$$

(b) For non-satiation $-\infty < w < -1/2d = 194.78$

Therefore utility function cannot help for wealth in excess of 194.78. Can not use the utility function.

12 Limitation of Utility Theory

1) Need to know the precise form and shape of the individual's utility function
2) The theorem cannot be applied separately to each of several sets of risky choices facing an individual.
3) For corporate risk management, it may not be possible to consider a utility function for the firm as though the firm was an individual.

13 Construction of Utility Functions

By direct questioning

- Ask the individual what his/her utility function is
- Unlikely to work, as it's difficult to state your utility function mathematically

by indirect questioning

- It involves two steps
- firstly, fixing two values of the utility function for the two extremes of wealth being considered
- Secondly, the individual is asked to identify a certain level of wealth such that he or she would be indifferent between that certain level of wealth and a gamble that yields either of the two extremes with particular probabilities.
- The process is repeated for various scenarios until a sufficient number of plots is found.

14 Maximizing Utility Through Insurance

Utility theory can be used to explain decisions such as purchasing insurance or buying a lottery ticket.

Finding the maximum premium which an individual is prepared to pay:

$$E[\,U(w - X)] = U(w - P)$$

Where:

w - initial level of wealth.

X – amount of loss

P – maximum premium

U(x) – Individual’s utility function

15 Finding The Minimum Premium

The *minimum insurance premium* Q which an insurer should be prepared to charge for insurance against a risk with potential loss *Y* is given by the solution of the equation:

$$E[\,U(w + Q - Y)] = U(w)$$

Where

w - initial wealth

U(w) – Insurer’s utility function

Q – min premium which an insurer will offer

Summary

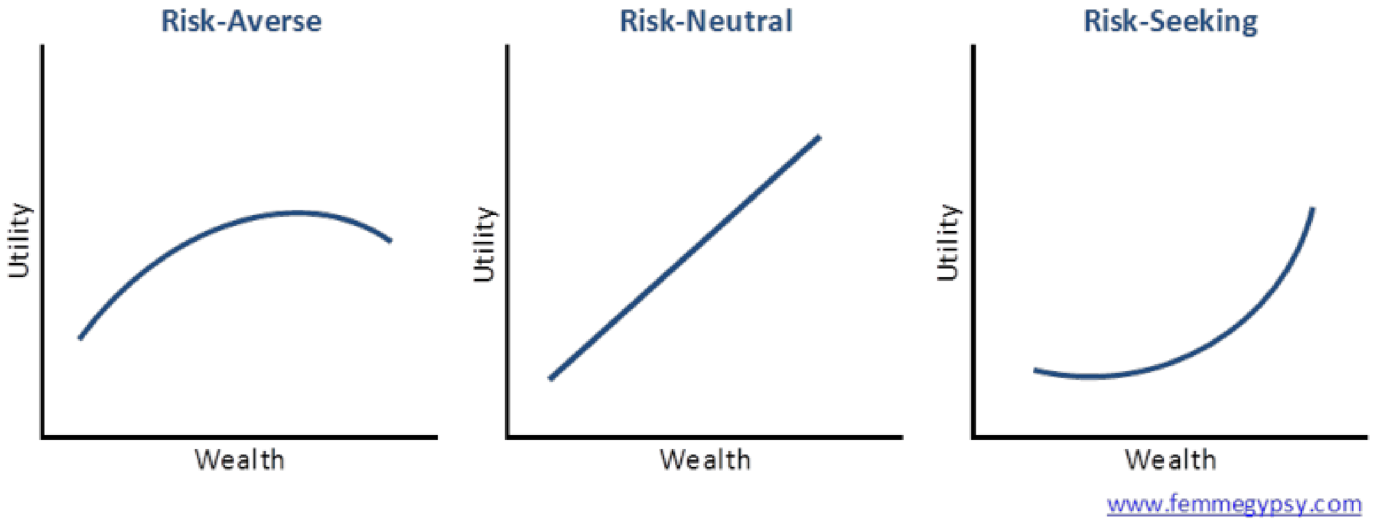
- Utility, in economics, refers to the usefulness or enjoyment a consumer can get from a service or good.
- Expected utility $E(U) = \sum_{i=1}^{n} p_i U(W_i)$
- Four Axioms for Expected utility theorem are –

1. Comparability
2. Transitivity
3. Independence
4. Certainty equivalence

- Non-satiation means people prefer more to less i.e. Never satisfied.
- Marginal utility of wealth= U'(W)>0
- A gamble which has an expected value of 0 is a fair gamble.

Summary

- Investors can be risk averse(hates risk), risk seeking(loves risk) or risk neutral(neutral to risk).
- Graphs-



- Types of gamble- Additive and Multiplicative
- Arrow-Pratt measure of risk aversion- Absolute risk aversion(ARA) and relative risk aversion(RRA).

Summary

- Types of Utility functions- Quadratic, Log and Power.

- An iso elastic utility function is one that exhibits constant RRA, i.e. $R'(w) = 0$

  Example-Log and power utility functions

- Utility function can be constructed by Direct questioning and by indirect questioning

- The *minimum insurance premium Q* which an insurer should be prepared to charge for insurance against a risk with potential loss *Y* is given by the solution of the equation:

$$E[U(w + Q) - Y] = U(w)$$

Question

CT8, September 2013, Q1

i. State the expected utility theorem
ii. State the four axioms from which it can be derived
iii. Explain the concepts of non satiation and risk aversion, showing how they can be expressed in terms of a utility function

A quadratic utility function is given by the equation $U(w) = w + bw^2$.
The value of absolute risk aversion at a value of wealth of one unit is 0.25.

i. Calculate the value of b and the range over which U(.) satisfies the condition of non satiation

Solution

Answer:

i)
(a) The expected utility theorem states that a function, U(w) can be constructed representing an investor's utility of wealth, w, at some future date. Decisions are made on the basis of maximising the expected value of utility under the investor's particular beliefs about the probability of different outcomes.

(b) The expected utility theorem can be derived formally from the following four axioms.

1. Comparability

An investor can state a preference between all available certain outcomes.

2. Transitivity

If A is preferred to B and B is preferred to C, then A is preferred to C.

Solution

3. Independence

If an investor is indifferent between two certain outcomes, A and B, then he is also indifferent between the following two gambles:

(a) A with probability p and C with probability (1 − p); and (b) B with probability p and C with probability (1 − p).

4. Certainty equivalence

Suppose that A is preferred to B and B is preferred to C. Then there is a unique probability, p, such that the investor is indifferent between B and a gamble giving A with probability p and C with probability (1 − p).

B is known as the certainty equivalent of the above gamble.

Solution

(ii) It is usually assumed that people prefer more wealth to less. This is known as the principle of non-satiation and can be expressed as: U ′(w)>0 or U is strictly increasing.

Attitudes to risk can also be expressed in terms of the properties of utility functions.

A risk averse investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. The utility function condition is U″(w)< 0 or U is strictly concave.

(iii) The absolute risk aversion A is given by: $A(w) = \frac{-U''(w)}{U'(w)}$.

Which for the utility function given can be calculated by taking derivatives as,

Now, given the condition A(1) = 0.25 yields b = ·0.1.

Non-satiation means : $U'(w) > 0 \iff 1+2bw > 0 \iff -\infty < w < 5$.

Question

CT8, November 2008 ,Q. 3

A risk-averse and non-satiated person has been offered a job by a highly reputed IT company. His fixed salary was Rs. 50 lakhs p.a. while he will be paid Rs. 30 lakhs as bonus if the company exceeds its annual growth target. The probability of company exceeding its growth target is:

Probability of exceeding growth target	GDP growth
100%	>10%
60%	5%-10%
0%	<5%

(1) The utility function of the person is $U(x) = 0.9x^2 - 0.4x^3$ (x represents a proportion of Rs. 80 lakhs). Derive a salary range according to the utility function.

(2) The expected utility offer by the job.

Solution

Answer:

(1) Probability of company exceeding its growth target : (33%)*(100%) + (33%)*(60%) + (33%)*(0%) = 53.3%

As the person is non-satiated, he will prefer more to less, therefore U'(x) > 0

Now $U(x) = 0.9x^2 - 0.4x^3$

Therefore, $U'(x) = 1.8x - 1.2\,x^2$

or $1.2x*(1.5 - x) > 0$

or the **boundary is** $\mathbf{x < 1.5}$

(2) Expected utility offered by the job :

(53.3%)*[(0.9*1^2) – (0.4*1^3)] + (1-53.3%)*[(0.9*(50/80)^2) – (0.4*(50/80)^3)] = 0.385

Question

CT8, November 2013 ,Q. 8

An investor has a quadratic utility function $U(w) = w + \alpha w^2$

i. Over what constraints does this satisfy the requirements of non – satiation and risk aversion?

ii. Using the Arrow –Pratt measures of risk aversion determine if this investor exhibits increasing or declining risk aversion.

iii. Show the expected utility of the investor can be expressed as a linear combination of only the first two moments of the distribution of wealth.

Solution

i. $U(w) = w+\alpha w^2 \Rightarrow U'(w) = 1+2\alpha w$ and $U''(w) = 2\alpha$

i. For $U(w)$ to satisfy the requirement of risk aversion $U''(w) = 2\alpha < 0$ i.e. $\alpha < 0$
For $U(w)$ to satisfy non-satiation $U'(w) = 1 + 2\alpha w > 0$ i.e. $-\infty < w < -1/2\alpha$ (where $\alpha < 0$)

(1 Mark)

ii. $A(w) = \frac{-U''(w)}{U'(w)} = \frac{-2\alpha}{1+2\alpha w}$ and $A'(w) = \frac{4\alpha^2}{(1+2\alpha w)^2} > 0$; thus increasing absolute risk aversion

$R(w) = w.\frac{-U''(w)}{U'(w)} = \frac{-2\alpha w}{1+2\alpha w}$ and $R'(w) = \frac{-2\alpha}{1+2\alpha w} + \frac{4\alpha^2 w}{(1+2\alpha w)^2} > 0$ (because $\alpha < 0$); thus implying increasing relative risk aversion

(2 Marks)

iii. The expected utility of this investor is $E[U(w)] = E[w+\alpha w^2] = E[w] + \alpha E[w^2]$ which is a linear combination of only the first two moments of the distribution of wealth.

Thank You