Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: SY BSc

Subject : Portfolio Theory and Security Analysis

Subject Code:

Chapter: Unit 1 Chapter 3

Chapter Name: Measures of Investment Risk

Today’s Agenda

1. Motivation
2. Variance
    1. Properties of Variance
3. Downside semi variance
4. Shortfall probability
5. Expected Shortfall
6. Conditional Expected Shortfall
7. Value at Risk
    1. VaR assumptions
    2. Advantages of VaR
    3. Useful Formulae on VaR
    4. Calculating VaR for Normal Distribution
    5. Why VaR is problematic?
    6. Shortcomings of VaR
8. Tail VaR

1 Motivation

Return is simple to measure.

When measuring risk there are several ways (or measures).

2 Variance (Var[X])

Continuous Distribution	Discrete Distribution
$E(X-\mu)^2 = \int_{all\ x} (x-\mu)^2 f(x)dx$	$\sum_{all\ x} (X-\mu)^2 P[X=x]$

- Where **X** is a random variable depicting Investment Return
- μ is the mean return

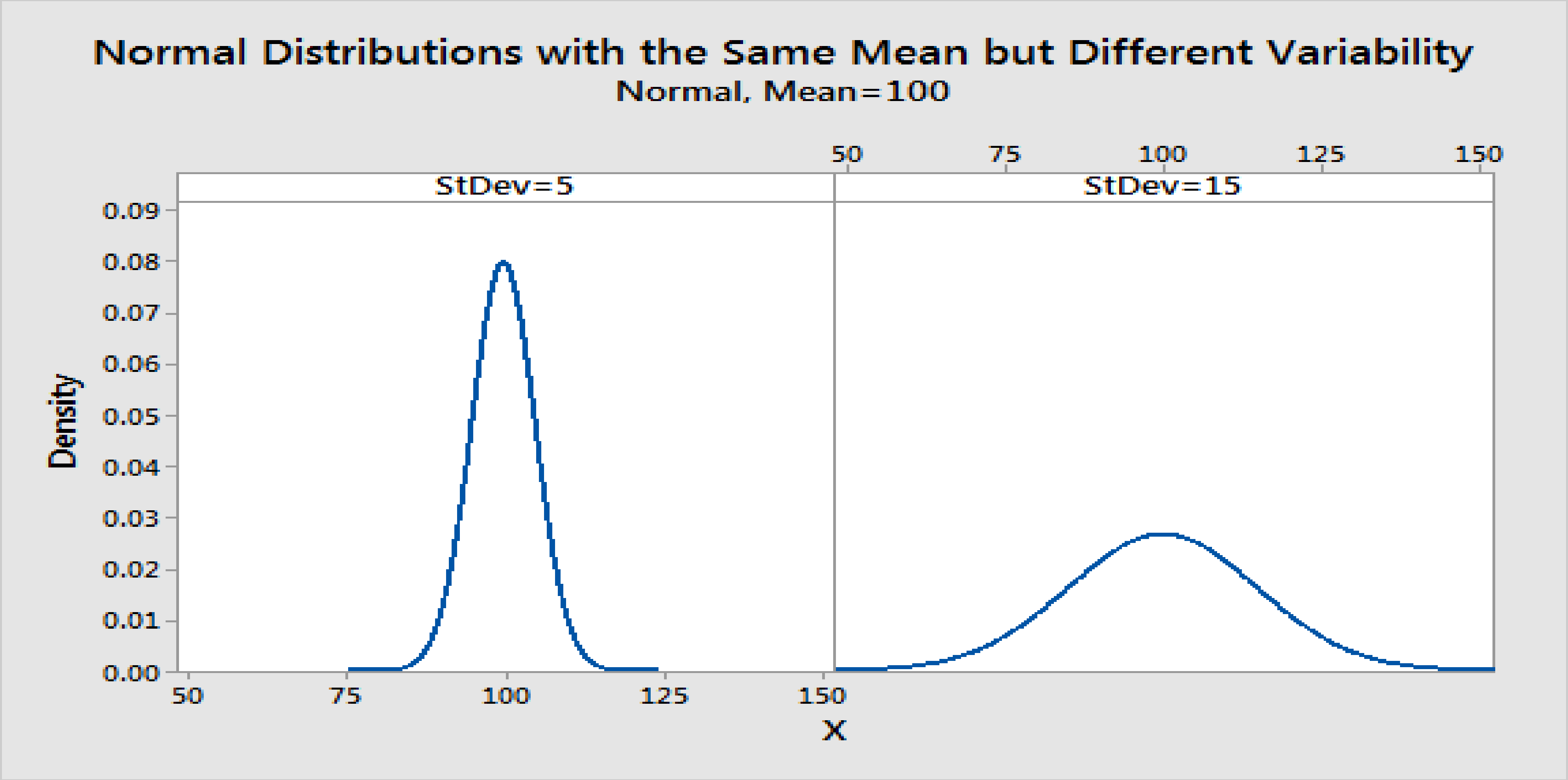
Pros

\+ Mathematically Tractable

Cons

\- Treats upside and downside risk equally

2 Variance (Var[X])



2.1 Properties of Variance

- If X and Y are two independent random variables then:

$$var(X+Y) = Var(X) + Var(Y)$$

- For constants a and b,

$$var(aX + b) = a^2 Var(X)$$

Question

CT8, May 2011, Q.9

You are contemplating an investment with a return of Rs X, where: X = 300,000 - 500,000U where U is a uniform [0, 1] random variable.

Calculate each of the following four measures of risk:

a) variance of return

Solution

Answer:

(i) $\text{Var}(X) = 500{,}000^2 * \text{Var}(U)$

$= 2.5 * 10^{11} * 1/12$

$= 2.08333 * 10^{10}$

3 Downside Semi-Variance

Continuous Distribution

$$DSV=\int_{x<\mu}(x-\mu)^2 f(x)dx$$

Discrete Distribution

$$DSV=\sum_{x<\mu}(x-\mu)^2 P[X=x]$$

Where μ is the mean return

Pros:

+The focus is on downside risk

Cons:

-Mathematically less tractable
-Ignores upside risk
-Measures downside risk relative to the mean, rather than a benchmark.

Question

CT8, May 2007 , Q. 11

Consider a zero-coupon corporate bond that promises to pay a return of 12% next period. Suppose that there is a 20% chance that the issuing company will default on the bond payment, in which case there is an equal chance of receiving a return of either 8% or 0%.

Calculate values for the following measures of investment risk:

(a) downside semi-variance

Solution

Answer:

Downside semi-variance The expected return on the bond is given by:
$0.80 \times 12\% + 0.10 \times 8\% + 0.10 \times 0\% = 10.4\%$

So the downside semi-variance is equal to:
$(10.4 - 8)^2 * 0.10 + (10.4 - 0)^2 \times 0.10 = 11.39\%\%$

Question

CT8, May 2011, Q.9

You are contemplating an investment with a return of Rs X, where: X = 300,000 - 500,000U where U is a uniform [0, 1] random variable.

Calculate each of the following four measures of risk:
a) downside semi-variance of return

Solution

Answer:

Downside semi-variance of X = 2.5 * 10^{11} * upside semi-variance of U;

the upside semi-variance of U is by symmetry 1/24

So,
downside semi-variance of X is 1.04166 *10^{10}

4 Shortfall Probability

Continuous Distribution	Discrete Distribution
SP $= \int_{x<L} f(x)dx$	SP $= \sum_{x<L} P[X = x]$

where:
L – chosen benchmark level

- SP measures the probability of returns falling below a certain benchmark level.
- Same as calculating cumulative DF at benchmark L :

$$P[\,X < L] = F_X(L)$$

4 Shortfall Probability

Pros:	***Cons:***
❑ Easy to understand	❑ No information about the extent of the shortfall
❑ Tractable	❑ Ignores upside risk
❑ Choice of benchmark level	

Question

CT8, April 2016 , Q.2

The returns on an asset follow a Normal distribution with mean μ = 6% per annum and variance σ2 = 23% per annum. An investor buys €500 of the asset.

i) Determine the shortfall probability for the value of the asset in one year's time below a value of €480.

ii) Explain what can be deduced about an investor's utility function if the investor makes decisions based on:
(b) the shortfall probability of returns.

Solution

Answer:

i) The shortfall probability required is the probability that the return is lower than 480/500 – 1 = –4% i.e.

$P(N(6\%, 23\%) \leq 4\%)$

$= P(Z \leq (-4\% - 6\%)/\sqrt{(23\%)})$

$= P(Z \leq -0.20851)$

$= 0.417$

ii)

(b) This corresponds to a utility function which has a discontinuity at the minimum required return.

Question

CT8, May 2011, Q.9

You are contemplating an investment with a return of Rs X, where: X = 300,000 - 500,000U where U is a uniform [0, 1] random variable.

Calculate each of the following four measures of risk:
a) shortfall probability, where the shortfall level is Rs 100,000

Solution

Answer:

$$P(X < 100{,}000) = P(U > 0.4) \\ = 0.6$$

5 Expected Shortfall; E[max(L-X,0)]

Continuous Distribution

$$ES = \int_{x<L} (L-x) f x \, dx$$

Discrete Distribution

$$ES = \sum_{x<L} (L-x) P[X = x]$$

where:
L – chosen *benchmark level*

Same as calculating E[max(L-X,0)]

6 Conditional Expected Shortfall

Continuous Distribution

CES = (∫ $(L\text{-x}) f x \mathrm{d}x$)/P[X<L]
x<L

Discrete Distribution

CES = $(\sum_{x<L}(L-x)P[X=x])/P[X=x]$

where:
L–chosen*benchmarklevel*

Analogoustocalculating:E[L–X|X<L]

Question

CT8, October 2016, Q.1

A farmer has a small apple tree which produces one harvest of apples per year. The number of apples the tree produces follows a Poisson distribution with a mean and variance of 8.
Determine the expected shortfall below a harvest of 5 apples

Solution

Answer:

$P(X = 0) \times (5 - 0) = 0.002$
$P(X = 1) \times (5 - 1) = 0.011$
$P(X = 2) \times (5 - 2) = 0.032$
$P(X = 3) \times (5 - 3) = 0.057$
$P(X = 4) \times (5 - 4) = 0.057$

Summing the above, we get 0.159

So the expected shortfall below 5 apples is 0.159 apples

Question

CT8, May 2007, Q.11

Consider a zero-coupon corporate bond that promises to pay a return of 12% next period. Suppose that there is a 20% chance that the issuing company will default on the bond payment, in which case there is an equal chance of receiving a return of either 8% or 0%.

Calculate values for the following measures of investment risk:

a) shortfall probability based on the risk-free rate of return of 8.5%

b) the expected shortfall below the risk-free return conditional on a shortfall occurring.

Solution

Answer:

a) Shortfall probability

The probability of receiving less than 8.5% is equal to the sum of the probabilities of receiving 8% and 0%, ie 0.20.

b) Expected conditional shortfall

The expected shortfall below the risk-free rate of 8.5% is given by:

$(8.5 – 8) \times 0.10 + (8.5 – 0) \times 0.10 = 0.90\%$

The expected shortfall below the risk-free return conditional on a shortfall occurring is equal to:

Expected shortfall/shortfall probability = 0.90%/0.2 = 4.5%

7 Value at Risk (VaR)

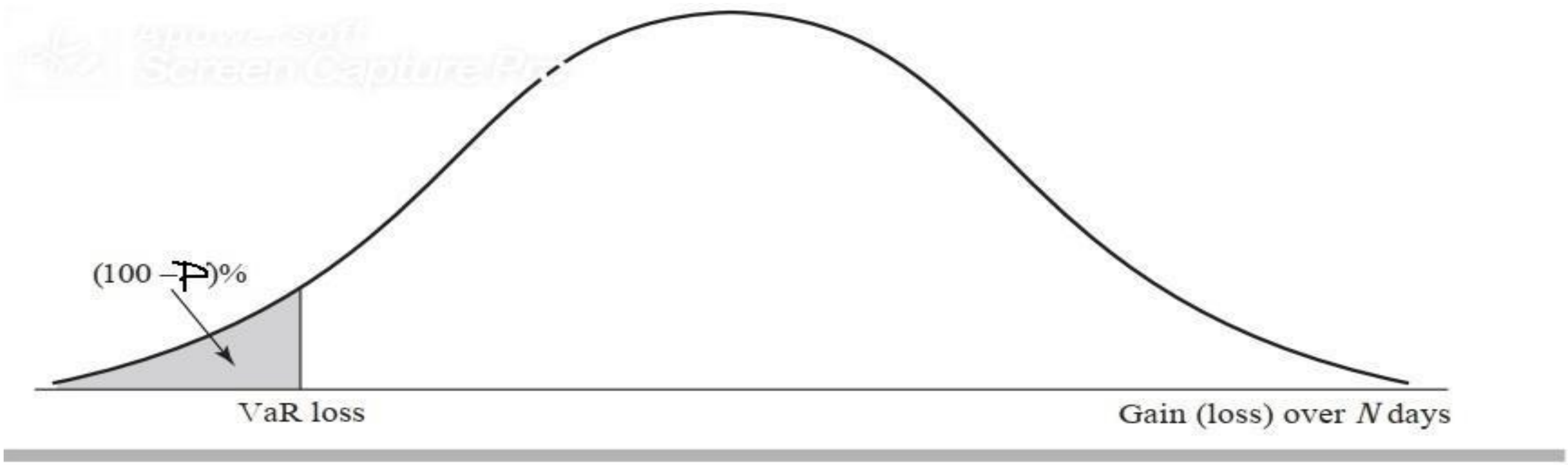
The Question Being Asked in VaR:

- "What loss level is such that we are *P*% confident it will not be exceeded in *N* business days?"
- For e.g. " we are 95% confident that we will not loose >$1M over next 5 days
- VaR is a function of two parameters: the time horizon (N days) and the confidence level (P%)

7 Value at Risk (VaR)

VaR is the loss level that will not be exceeded with a specified probability.

Probability (Inv. Return < VaR Loss) = Confidence Level (100- P)%



7.1 VaR Assumptions

Assumptions underlying calculation of VaR:

- Often returns are assumed to be normally distributed.
- In practice, returns may be skewed (or fat – tailed)

Question (Discrete)

CT8, October 2016, Q.1

A farmer has a small apple tree which produces one harvest of apples per year. The number of apples the tree produces follows a Poisson distribution with a mean and variance of 8.

Determine the 10% Value at Risk level for the number of apples produced.

Solution

Answer:

$P(X = 0) = (8^0 e^{-8})/0! = 0.00034$
$P(X = 1) = (8^1 e^{-8})/1! = 0.00268$
$P(X = 2) = (8^2 e^{-8})/2! = 0.01073$
$P(X = 3) = (8^3 e^{-8})/3! = 0.02863$
$P(X = 4) = (8^4 e^{-8})/4! = 0.05725$

So $P(X \leq 4) = 0.00034 + 0.00268 + 0.01073 + 0.02863 + 0.05725 = 0.09963$

Alternatively, directly from the Formulae & Tables: $P(X \leq 4) = 0.09963$ [1]

$P(X = 5) = (8^4 e^{-8})/5! = 0.09160$ So $P(X \leq 5) = 0.191236$ (or directly from the Formulae & Tables)

So the 10% VaR level is 5 (or –5) apples.

Question (Continuous)

CT8, April 2015,Q.2

Assume X has a Normal distribution with mean μ = 5% and variance $\sigma2$ = 100%%.
(iii) Calculate the 5% Value at Risk.

Solution

Answer:

$P(X < t) = 5\%$ where $x \sim N(5, 100)$
$\Leftrightarrow P(Z \leq (t-5) / 10) = 5\%$
$\Leftrightarrow (t-5) / 10 = -1.645$
$\Leftrightarrow t = -11.45\%$

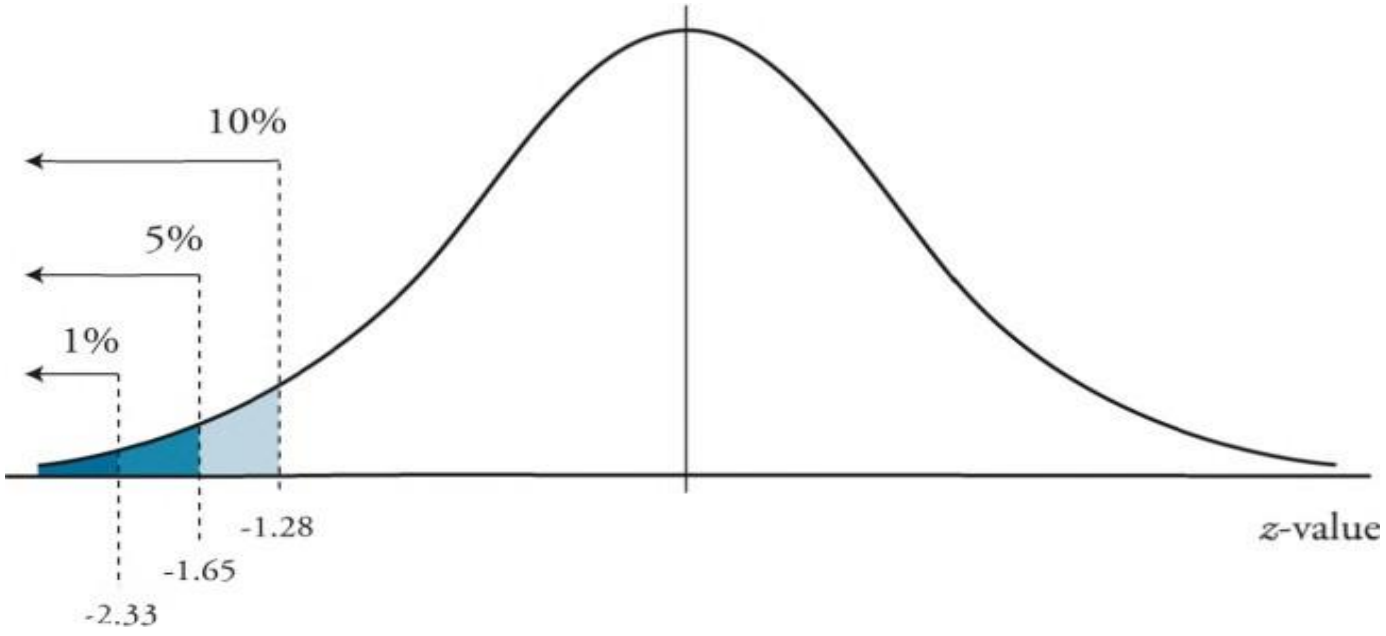
Therefore the 5% VaR is $-t = 11.45\%$.

7.2 Advantages of VaR

- VaR captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”
- Bank regulators also use VaR in determining the capital a bank is required to keep for the risks it is bearing.

7.3 Useful Formulae on VaR

$$VaR(X\%)_{dollar\ basis} = VaR(X\%)_{decimal\ basis}\ X\ asset\ (or\ portfolio)\ value$$



7.4 Calculating VaR for Normal Distribution

For normal distribution with mean μ and standard deviation σ

$$\text{VaR}(X\%) = [\mu + z_{(100-x)\%}\ \sigma]$$

Where,

- VaR(X%) – the X% probability value at risk
- $z_{(100-x)\%}$ - the critical z-value based on the normal distribution and the selected x% probability
- σ – standard deviation

7.5 Why VaR is Problematic?

- Consider an investment bank, where a VaR limit (confidence level 99%) of say $50,000 is
- imposed on a certain derivatives trader.

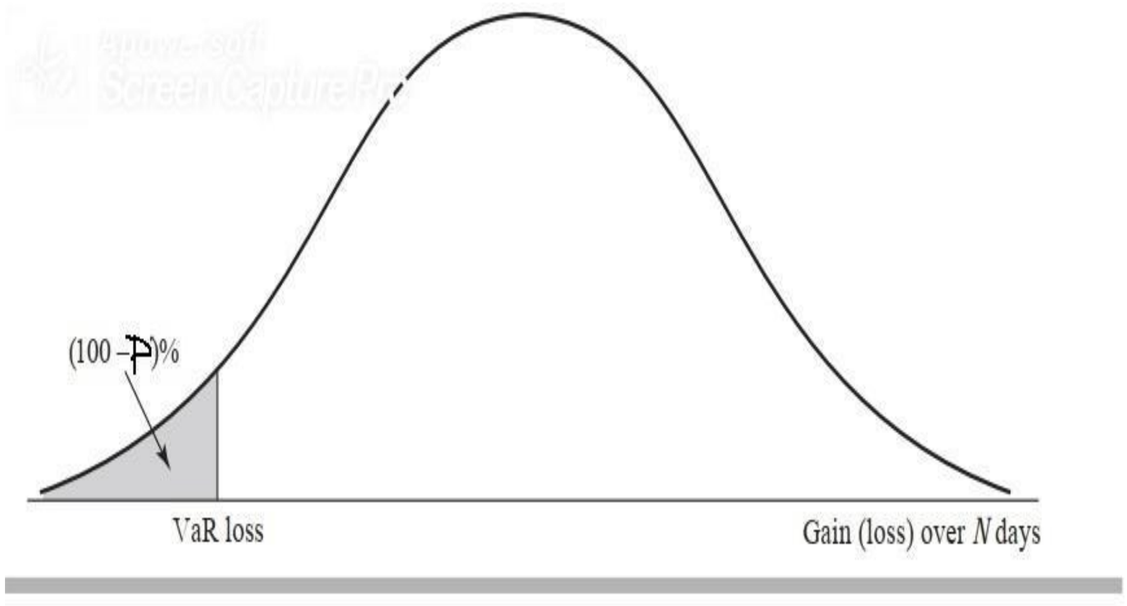
- The meaning of this is that a loss of more than $50,000 should occur only once in every hundred trading days on average.

- But because of the very definition of VaR, there is no differentiation between small and very large violations of the $50,000 limit.

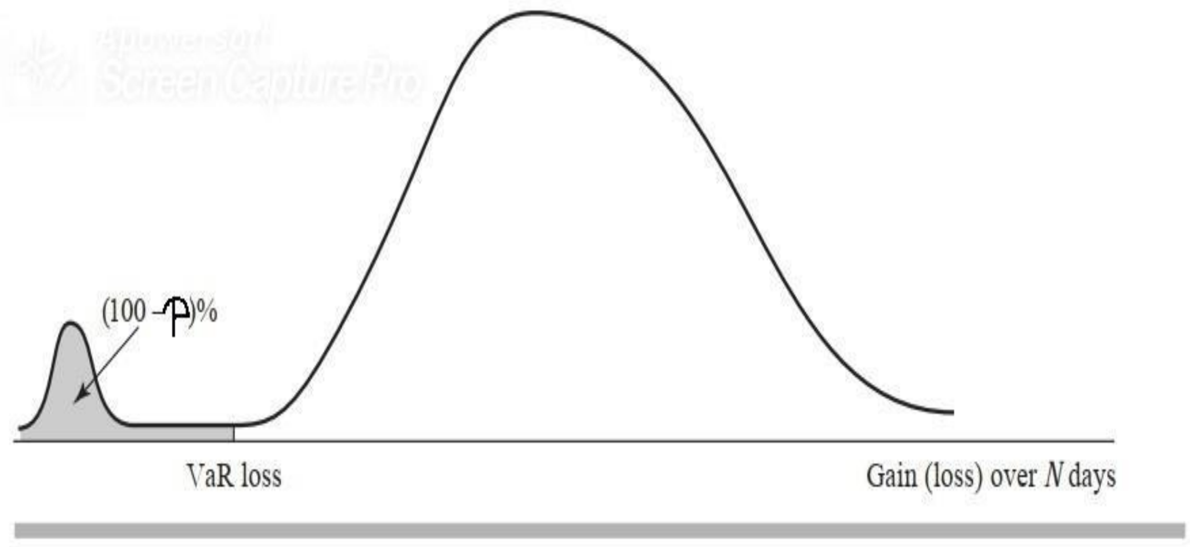
- The eventual loss could be $60,000 as well as $600,000

7.6 Shortcomings of VaR

VaR level at P%



Alternative Situation.
VaR is the same, but the potential loss is larger.



8 Tail VaR

- Tail VaR answers the question: "*If things do get bad, just how bad will they be.*"
- Same as Expected Shortfall
- Conditional Tail VaR is same as conditional expected shortfall
- Calculated as E[max(L-X,0)]
- **If *L* is chosen to be a particular percentile point on the distribution, then the risk measure is known as the Tail-VaR.**

Question

CT8, May 2012, Q.3

Investment returns (% pa), X , on a particular asset are modelled using the probability distribution:

X	Probability
-10	0.1
5.5	0.9

For a portfolio consisting of INR 20 crores invested in the asset, calculate the following

(i) Mean
(ii) Variance
(iii) 95% VaR over one year
(iv) 95% TailVaR over one year.

Solution

Answer:

i) Mean is given by =-10*0.1+5.5*0.9 = 3.95

ii) Variance = $(3.95-(-10))^2 * 0.1 + (3.95 - 5.5)^2 * 0.9 = 21.62$

iii) 95% Value at Risk at
VaR(X) = -t where t = max { x : P(X<x) ≤ 0.05}
P(X<-10) = 0 and P(X< 5.5)= 0.1
t = -10

Since t is a percentage investment return per annum, the 95% value at risk over one year on a Rs. 20 crores portfolio is 20x0.10 = Rs. 2 crores. This means that we are 95% certain that we will not make profit of less than Rs. 2 crores over the next year.

Solution

iv) The expected shortfall in returns below -10% is given by

$$E(\min(-10-X,0)) = \Sigma\,(-10 - x)P(X=x) \qquad x<-10$$
$$= 0$$

On a portfolio of Rs. 20 crores, the 95% TailVaR = 0.
This means expected reduction in profit below Rs. -2 crores is zero. That is, profit can not fall below Rs. -2 crores.

Thank You