Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

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Chapter: Unit 3 Chapter 3

Chapter Name: Capital Asset Pricing Model

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1 Introduction

- The **Capital Asset Pricing Model (CAPM)** describes the relationship between the expected return of assets and the systematic risk of the market.

- CAPM indicates that the expected return of an asset is equal to the risk-free return plus a risk premium.

- The assumption of CAPM is that investors are rational and want to maximize return and reduce risk as much as possible.

- The goal of CAPM is thus to calculate what return an investor can expect to make for a given risk premium over the risk-free rate.

1 Introduction

- The standard form of the general equilibrium relationship for asset returns was developed independently by Sharpe, Lintner, and Mossin. Hence it is often referred to as the Sharpe–Lintner–Mossin form of the capital asset pricing model.

- This model has been derived in several forms involving different degrees of rigor and mathematical complexity.

- There is a trade-off between these derivations.

- The more complex forms are more rigorous and provide a framework within which alternative sets of assumptions can be examined.

- However, because of their complexity, they do not convey the economic intuition behind the CAPM as readily as some of the simpler forms. Because of this, we approach the derivation of the model at two distinct levels. The first derivation consists of a simple, intuitively appealing derivation of the CAPM. This is followed by a more rigorous derivation.

1 Discuss

What are the assumptions of MVPT?

1.1 Assumptions of MVPT

- All expected returns, variances and covariances of pairs of assets are known.
- Investors make their decisions purely on the basis of expected return and variance.
- Investors are non-satiated.
- Investors are risk-averse.
- There is a fixed single-step time period.
- There are no taxes or transaction costs.
- Assets may be held in any amounts, (with short-selling, infinitely divisible holdings, no maximum investment limits).

1.2 Extra Assumptions of CAPM

- All investors have the same one-period horizon.
- All investors can borrow or lend unlimited amounts at the same risk-free rate.
- The markets for risky assets are perfect. Information is freely and instantly available to all investors and no investor believes that they can affect the price of a security by their own actions.
- Investors have the same estimates of the expected returns, standard deviations and covariances of securities over the one-period horizon.
- All investors measure in the same 'currency' *eg:* pounds or dollars or in 'real' or 'money' terms.

1.3 Consequences of the extra assumptions

- If investors have homogeneous expectations, then they are all faced with the same efficient frontier of risky securities

- If in addition they are all subject to the same risk-free rate of interest, the efficient frontier collapses to the straight line in E-σ space which passes through the risk-free rate of return on the E- axis and is tangential to the efficient frontier for risky securities.

2 Market Portfolio

The market capitalization of a company is the total value of its shares, ie the number of shares issued multiplied by the share price ($N_i P_i$).

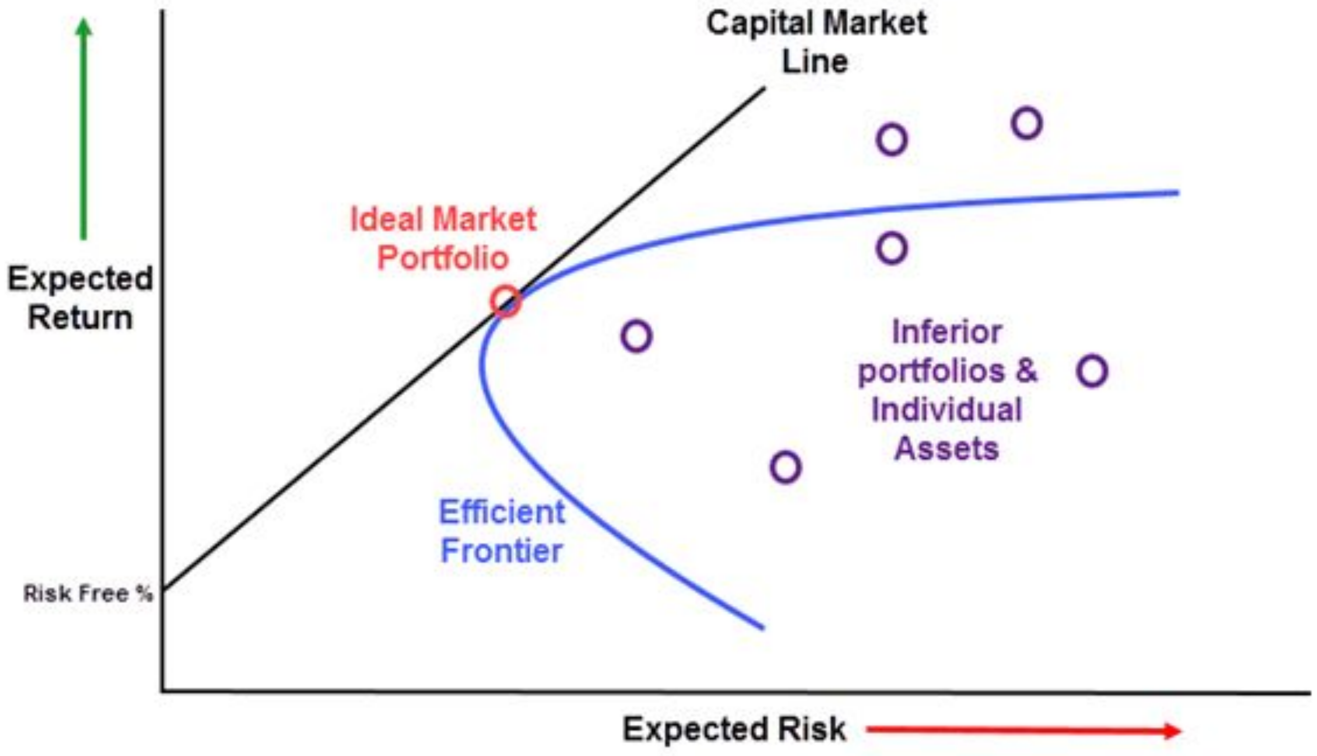
The market portfolio consists of all risky assets held in proportion to their market capitalization, ie the portfolio weighting of the share i is:

$$w_i = \frac{N_i P_i}{\sum_i N_i P_i} = \frac{\text{market value of the individual asset}}{\text{market value of all assets}}$$

In expected return-standard deviation space, the market portfolio is situated at the point where a straight line through the risk-free return is tangential to the efficient frontier of risky assets only.

2.1 CAPM Prediction about Investment Choice

CAPM predicts that all rational investors will hold a combination of the risk free asset and the market portfolio of risky assets.



2.2 Will all investors have the same portfolio under CAPM?

No, each investor will have a different portfolio, reflecting his individual risk-return preferences.

However, each investor's optimal portfolio will consist of some combination of the risk-free asset and the market portfolio.

3 Separation Theorem

The proposition that the investment decision, which involves investing in the market portfolio on the capital market line, is separate from the financing decision, which targets a specific point on the CML based on the investor's risk preference.

The optimal combination of risky assets is the market portfolio.

3.1 Two Fund Separation

Each investor will have a utility-maximizing portfolio that is a combination of the risk-free asset and a portfolio (or fund) of risky assets that is determined by the line drawn from the risk-free rate of return tangent to the investor's efficient set of risky assets.

This line has come to be known as the **Capital Market Line**. It represents a linear relationship between portfolio risk and return.

4 Capital Market Line

The capital market line is the straight line in expected return-standard deviation space representing the efficient frontier. It has the equation:

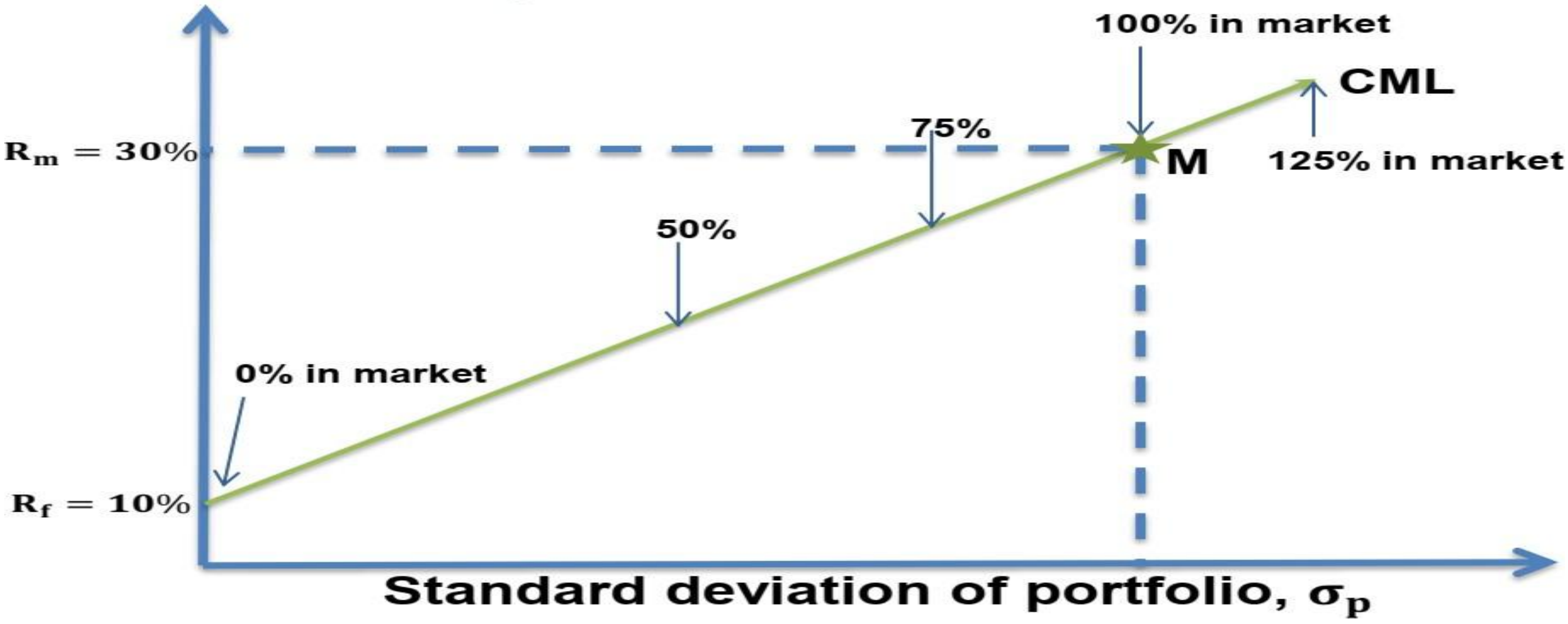
$$E_p - r_f = \frac{(E_M - r_f)}{\sigma_M} \sigma_p$$

All efficient portfolios lie on the **capital market line**.

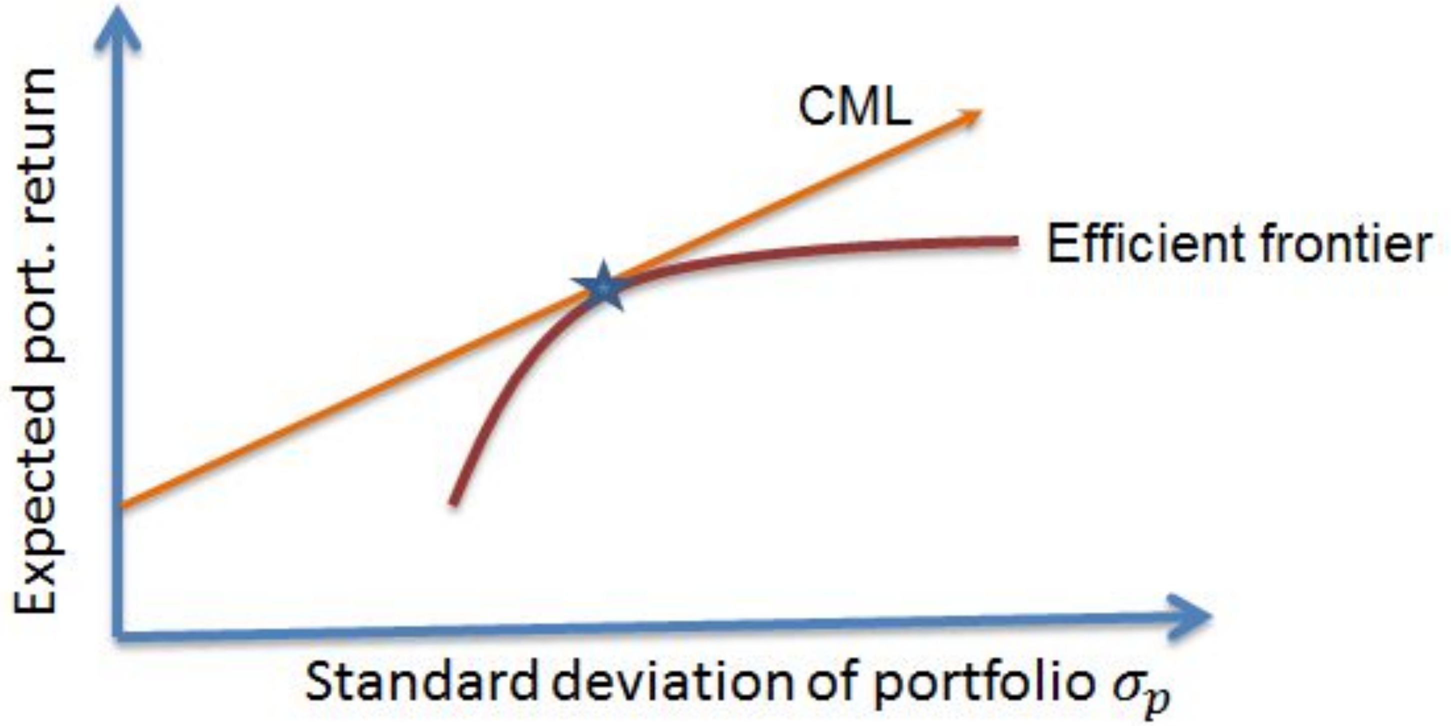
This equation appears on the Page 43 of the Tables.

4 Capital Market Line

The capital market line expresses the expected return of a portfolio as a linear function of the risk-free rate, the portfolio's standard deviation, and the market portfolio's return and standard deviation.



4 Capital Market Line



4.1 Market price of risk

The market price of risk is the excess expected return on the market portfolio per unit of risk, the market risk premium per unit of volatility.

In the CAPM, it is equal to $(E_M - r_f)/\sigma_M$

$(E_M - r_f)$ is the market risk premium.

4.2 Security Market Line

The security market line is the straight line in the expected return– beta space on which all securities and portfolios lie.

It has the equation:

$$E_i - r_f = \beta_i(E_M - r_f)$$

This equation appears on the page 43 of the Tables.

5 Limitations of CAPM

1. It is based on unrealistic assumptions.

2. It is difficult to test. In particular, the market portfolio is difficult to identify because of the wide range of the available instruments, including equities, bonds, cash, property, commodities and human capital.

3. Empirical studies do not provide strong support for the model.

4. Other versions of the basic CAPM have been produced which allow for taxes, inflation, and also for a situation where there is no riskless asset. In the international situation there is no asset which is riskless for all investors (due to currency risks) so a model has been developed which allows for groups of investors in different countries, each of which considers their domestic currency to be risk-free.

6 Extensions of the basic CAPM

Models have been developed which allow for decisions over multiple periods and for the optimization of consumption over time to take account of this.

Other versions of the basic CAPM have been produced which allow for taxes and inflation, and also for a situation where there is no riskless asset. Examples of these models include:

1. Multi-period models
2. Model with taxes
3. Zero-beta model

Summary

- The Capital Asset Pricing Model (CAPM) describes the relationship between the expected return of assets and the systematic risk of the market.

- CAPM indicates that the expected return of an asset is equal to the risk-free return plus a risk premium,

- Assumptions:.
  Investors make their decisions purely on the basis of expected return and variance.
  Investors are non-satiated.
  Investors are risk-averse.

- The fact that the optimal combination of risky assets for an investor can be determined without any knowledge of their preferences towards risk and return (or their liabilities) is known as the separation theorem.

- Capital Market Line is a linear relationship between portfolio risk and return.

Summary

- The capital market line is the straight line in expected return-standard deviation spacerepresenting the efficient frontier. It has the equation:

$$E_P - r_f = \frac{E_M - r_f}{\sigma_M} \sigma_P$$

- The market price of risk is the excess expected return on the market portfolio per unit of risk, the market risk premium per unit of volatility.
  $(E_M - r_f)$is the market risk premium

- The security market line is the straight line in the expected return- beta space on which allsecurities and portfolios lie. It has the equation:

$$E_i - r_f = \beta_i (E_M - r_f)$$

Questions

CT8, April 2013, Q4

In a market where the CAPM holds there are five assets with the following attributes.

Asset	A	B	C	D	E	***Probability of being in state***
Annual return in						
State 1	3%	3%	3%	3%	3%	0.25
State 2	5%	7%	2%	8%	3%	0.5
State 3	7%	5%	8%	1%	3%	0.25
Market Capitalisation	10m	20m	40m	30m		

(i) Calculate the expected annual return on the market portfolio and σ_M, the standard deviation of the annual return on the market portfolio.
(ii) Calculate the market price of risk under CAPM.
(iii) Calculate the beta of each asset.
(iv) Outline the limitations of the CAPM.

Solution

(i) The market portfolio is in proportion to the market capitalisation since every investor holds risky assets in proportion to that portfolio. Thus the market portfolio is 0.1A + 0.2B + 0.4C + 0.3D (asset E is the risk-free asset).

Asset	*A*	*B*	*C*	*D*	*E*	*Probability of being in state*
Annual return in						
State 1	3%	3%	3%	3%	3%	0.25
State 2	5%	7%	2%	8%	3%	0.5
State 3	7%	5%	8%	1%	3%	0.25
Market Capitalisation	10m	20m	40m	30m		

E_A = 5%; E_B = 5.5%; E_C = 3.75%; E_D = 5%

and so E_M = (10×5%+20×5.5%+40×3.75%+30×5%)/100 = 4.6%

Now σ_M^2 = 0.25×(3−4.6)2+0.5×(5.1−4.6)2+0.25×(5.2−4.6)2 = 0.855% and σ_M = 0.92466%

(ii) market price of risk is $(E_M - r)/\sigma_M$ = (4.6 − 3)/0.92466 = 173%

Solution

(iii) β_i = Cov(R_i, R_M) / Var(R_M).

Now,
Cov (R_A , R_M) = $0.25\times3\times3+0.5\times5\times5.1+0.25\times7\times5.2-5\times4.6$ = 1.1%;
Cov (R_B , R_M) = $0.25\times3\times3+0.5\times7\times5.1+0.25\times5\times5.2-5.5\times4.6$ = 1.3%;
Cov (R_C , R_M) = $0.25\times3\times3+0.5\times2\times5.1+0.25\times8\times5.2-3.75\times4.6$ = 0.5%;
Cov (R_D , R_M) = $0.25\times3\times3+0.5\times8\times5.1+0.25\times1\times5.2-5\times4.6$ = 0.95%

It follows that β_A = 1.1/0.855=1.2865, β_B = 1.3/0.855 = 1.5205, β_C = 0.5/0.855 = 0.5848 and β_D = 0.95/0.855 = 1.1111 OR
Assets all lie on the securities market line, so $E_i - r = \beta_i (E_M - r)$, so β_A = 2/1.6=1.25, β_B = 2.5/1.6 = 1.5625 β_C = 0.75/1.6 = 0.46875 and β_D = 2/1.6 = 1.25

(iv) Most of the assumptions of the basic model can be attacked as unrealistic. Empirical studies do not provide strong support for the model. There are basic problems in testing the model since, in theory, account has to be taken of the entire investment universe open to investors, not just capital markets.

Questions

CT8, September 2009, Q4

An investor invests a proportion x_i of the assets in his portfolio in the ith of N securities.

(i) State the expected return and variance of his portfolio. Define any notation you use.

Securities with the properties in the table below are available to an investor. The statistics in the table refer to the next year.

	A	*B*
Expected return	4%	3%
Variance of return	16%%	4%%

Correlation coefficient between assets $\rho_{AB} = 1$

The investor combines the securities to form a portfolio.

(ii) Calculate the relative amount which should be invested in each security to give a portfolio with the minimum possible variance. (Note: you may assume that short selling securities is allowable.)

(iii) Show that if it is possible to borrow at the rate of 1% p.a. over the next year, it is possible for the investor to make a risk free profit over the year without using any of his own capital.

Solution

(i) Expected Return = $\Sigma_i x_i E_i$
where Ei is expected return on security i.

Variance is $\Sigma_i \Sigma_j x_i x_j C_{ij}$
where C_{ij} is the covariance of the returns on securities i and j and $C_{ii} = V_i$
where V_i is variance of security i.

(ii) Proportion in A = $(V_B - C_{AB}) / (V_A + V_B - 2C_{AB})$

From (Unit 2 page 3) or can fairly easily be calculated from first principles

$C_{AB} = 1 * sd_A * sd_B = 1 * 4\% * 2\% = 8\%\%$

Thus Proportion in A = (4%% – 8%%) / (16%% + 4%% – 2 * 8%%) = –1
Proportion in B = 2
i.e. Short sell a unit of A and buy 2 of B.

Solution

(iii) Expected Return of portfolio in (ii) is -1 * 4 + 2 * 3 = 2%

Variance = 1^2 * 16%% + 2^2 * 4%% + 2* 2 * -1 * 2% * 4% = 0   (i.e. risk free)

Now if we borrow at 1% p.a. can invest in the portfolio in (ii) make a return of 2% pay back the loan and will have make 1% over the year.

Questions

CT8, Oct 2012 Q9

Consider a market where there are two risky assets A and B and a risk free asset. Both risky assets have the same market capitalization.
Assume that all the assumptions of the CAPM hold.
(i) State the composition of the market portfolio.
(ii) Derive the expressions for the variance of the market portfolio and for the beta of each asset, in terms of the variance of each asset and of their covariance.

Assume now that the risk-free rate is r_f = 10%, the expected return of the market portfolio is r_M =18%, the variance of asset A is 4%, the variance of asset B is 2% and their covariance is 1%.
(iii) Derive the value for the expected return on asset A and asset B.

An investor wants an expected return of 20%.
(iv) Calculate the composition of the corresponding portfolio.
(v) Derive the corresponding standard deviation using the Capital Market Line

Solution

(i) Since equal market capitalisation: $w_A = 0.5$ and $w_B = 0.5$.

(ii) Let r_M denote the return of the market portfolio, r_A (resp. r_B) denote the return of asset A (resp. asset B).

Then, $V(r_M) = V(0.5r_A + 0.5r_B)$
$= 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$.

$\text{Beta}_A = \text{cov}(r_A, r_M)/V(r_M)$
$= (0.5 * V(r_A) + 0.5 * \text{cov}(r_A, r_B)) / 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$.

As $\text{cov}(r_A, r_M) = \text{cov}(r_A, 0.5r_A + 0.5r_B)$
$= 0.5 * V(r_A) + 0.5 * \text{cov}(r_A, r_B)$

Similarly, $\text{Beta}_B = (0.5 * V(r_B) + 0.5 * \text{cov}(r_A, r_B)) / 0.5^2 * V(r_A) + 0.5^2 * V(r_B) + 2 * 0.5^2 \text{cov}(r_A, r_B)$

Solution

(iii) The equation of the Security Market line gives:

$r_i = r_f + Beta_i\,(r_M - r_f)$ where r_i is the expected return of asset i (for i = A,B).

Hence, using the numerical values, we get

$r_A = 0.2$ and $r_B = 0.16$

(iv) Using the separation theorem, we have:

$r_p = w_0\ rf + w_M\, r_M$

where w_0 is the weight of the risk-free asset in the portfolio P and w_M is the weight of the market portfolio in the portfolio P.

Moreover, there is the constraint $w_0 + w_M = 1$

Solving the system leads to:

$w_0 = -0.25$ and $w_M = 1.25$

Solution

(v) The Capital Market Line equation is:

$r_P = r_f + sigma_P * ((r_M – r_f) / sigma_M)$

where $sigma_P$ (resp. $sigma_M$) is the standard deviation of the portfolio P (resp. the market portfolio.

So,

we get $sigma_P = 17.6\%$

Thank You