INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical & Risk Modelling - 2

Chapter: Unit - 3

Category: Practice Questions

1. Subject CT4 April 2005 Question A5

A no chain Discount system operated by a motor insurer has the following four levels :

Level 1 : 0% discount

Level 2 : 15% discount

Level 3 : 40% discount

Level 4 : 60% discount

The rules for moving between these levels are as follows :

- Following a year with no claims, move to next higher level, of remain at level 4
- Following a year with no claims, move to next higher level, of remain at level 1
- Following a year with two or more claims, move back to levels, or move to level 1 (from level 2) or remain at level 1.

For a given policyholder the probability of no claims in a given year is 0.85 and the probability of making one claim is 0.12.

X(t) denotes the level of the policyholders in year t .

(i) (a) explain why X(t) is a Markov chain.

(b) Write down the transition matrix of this chain.

(ii) Calculate the probability that a policyholder who is currently at level 2 will be at level 2 after :

(a) One year

(b) Two year

(c) Three year

(iii) Explain whether the chains is irreducible and/or aperiodic.

(iv) Calculate the long-run probability that a policyholder is in discount level 2

2. Subject CT4 September 2006 Question A1

A manufacturer uses a test rig to estimate the failure rate in a batch of electronic components. The big holds 100 components and is designed to detect when a components fails, at which point it immediately replaces the component with another from the batch. The following are recorded for each of the n components used in the test (i = 1,2,. . . ,n).

S_i = time at which components I placed on the rig

T_i time at which component I removed of test period

$$f_i = \begin{cases} 1 & Component\ removed\ due\ to\ failuer \\ 0 & Component\ working\ at\ end\ of\ test\ period \end{cases}$$

The test rig was fully loaded and was run two years continuously.

You should assume that the force of failure, π of a component is constant and component failures are independent.

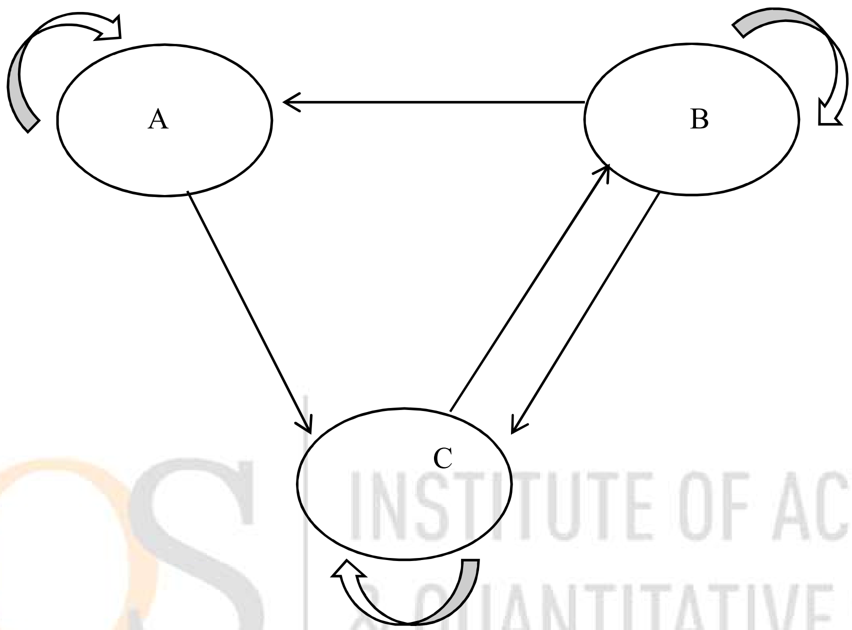
(i) Show that the contribution to the likelihood from component i is :

$\exp(-\mu\ (t_i - s_i))\ \mu^{f}$

(ii) Derive the maximum likelihood estimator for μ .

3. Subject CT4 April 2007 Question 6

A three state process with state space {A, B, C} is believed to follow a Markov chain with the following possible transitions :
An instrument



An instrument was used to monitor this process, but it was set up incorrectly and only recorded the state occupied after every two time periods. From these observations the following two-step transition probabilities have been estimated :

$P^2_{AA} = 0.5625$
$P^2_{AB} = 0.125$
$P^2_{BA} = 0.475$
$P^2_{CC} = 0.4$

Calculate the one step transition matrix consistent with these estimates.

4. Subject CT4 September 2008 Question 11

Consider the random variable defined by $X_n = \sum_{i=1}^{n} Y_j$ with each Y_j mutually independent with probability:

$P[Y_j = 1] = p$. $P[Y_j = -1] = 1 - p$ $0<p<1$

(i) Write down the state space and transition graph of the sequence X_n.

(ii) State , with reasons whether the process :

(a) Is aperiodic

(b) Is reducible

(c) Admits a stationary distribution.

Consider j > I > 0.

(iii) Derive an expression for the number of upward movements in the sequence X_n between t and (t + m) if $X_t = i$ and $X_{t+m} = j$.

(iv) Derive expressions for the m-step transition probabilities $P_{ij}^{(m)}$

(v) Show how the one-step transition probabilities would alter if X_n was restricted to non-negative numbers by introducing :

(a) A reflecting boundary at zero.

(b) An absorbing boundary at zero.

(vi) For each of the examples in part (v), explain whether the transition probabilities $p_{ij}^{(m)}$ would increase, decrease or stay the same. (Calculation of the transition probabilities is not required.)

5. Subject CT4 April 2009 Question 12

A motor insurer operates a no claims discount system with the following levels of discount {0%, 25%, 50%, 60%}.

The rules governing a policyholders discount level, based upon the number of claims made in the pervious year, are as follows :

- Following a year with no claims, the policyholders moves up one discount level, or remain at the 60% level.
- Following a year with no claims, the policyholders moves down one discount level, or remain at the 0% level.
- Following a year with two or claims, the policyholders moves down two discount level (subject to a limit of the 0% discount level).

The number of claims made by a policyholder in a year is assumed to follow a Poisson distribution with mean 0.30.

(i) Determine the transition matrix for the no claims discount system.
(ii) Calculate the stationary distribution of the system, π.
(iii) Calculate the expected average long-term level of discount.

The following data shows the number of the insurer's 130, 200 policyholders in the portfolio classified by the number of claims each policyholders made in the last year. This information was used to estimate the mean of 0.30.

No claims	96,632
One claim	28,648
Two claims	4,400
Three claims	476
Four claims	36
Five claims	8

(iv) Test the goodness of fit of these data to a Poisson distribution with mean 0.30.
(v) Comment on the implications of your conclusion in (iv) for the average level of discount applied.

6. **Subject CT4 April 2011 Question 2**

Distinguish between the conditional under which a Markov chain :

(a) Has at least one stationary distribution
(b) Ha unique stationary distribution
(c) Converges to a unique stationary distribution.

7. **Subject CT4 September 2012 Question 6**

(i) Define the stationary distribution of Markov chain.

A baseball stadium hosts a match each evening. As matches take place in the evening, floodlights are needed. The floodlights have a tendency to break down. If the floodlights break down, the game has to be abandoned and this costs this stadium $10,000. If the floodlights work throughout one match there is a 5% chance that they will fail and lead to the abandonment of the next match.

The stadium has an arrangement with the Flood watch repair company who are brought in the morning after a floodlight breakdown and charge $1,000 per day. There is a 60% chance they are able to repair the floodlights such that the evening game can take place and be completed without needing to be abandoned, if they are still broken the repair company is used (and paid) again each day until the lights are fixed, with the same 60% chance of fixing the light each day.

(ii) Write down the transition matrix for the process which describes whether the floodlights are working or not.
(iii) Derive the long run proportion of games which have to be abandoned.

The stadium manager is unhappy with the number of games being abandoned, and contacts the light Fantastic repair company who are estimate to have an 80% chance of repairing floodlights each day. However Light Fantastic will charge more than Flood watch.

(iv) Calculate the maximum amount the stadium should be prepared to pay Light Fantastic to improve profitability.

8. Subject CT4 April 2013 Question 11

(i) Explain what is meant by time-inhomogeneous Markov chain and give an example of one.

A no claims discount system is operated by a car insurer. There are four levels of discount : 0%, 10%, 25, and 40%. After a claim free year a policyholder moves up one level (or remains at the level 40% level). If a policyholder makes one claim in a year he or she moves down one level (or remain at the 0% level). A policyholder who makes more than one claim in one year moves down two levels (or moves to or remains at the 0% level). Changes in level can only happen at the end of each year.

(ii) Describe giving an example the nature of the boundaries of this process.

(iii) (a) state how many states are required to model this as a Markov chain.

(b) draw the transition graph.

The probability of a claim in any given months is assumed to be constant at 0.04. At most one claim can be made per month and claims are independent.

(iv) Calculate the proportion of policyholders in the long run who are at the 25% level.

(v) Discuss the appropriateness of the model.

9. Subject 103 April 2004 Question 7

i) Suppose that $\{X_t : t \geq 0\}$ is a time-homogeneous Markov jump process with generator matrix A and transition matrix P(t). Let π be a (column) vector of probabilities such that $\pi^T A = A^T$, Where 0 is a (column) vector whose components are all equal to zero.

a) Prove that

$$\frac{d}{dt}\pi^T P(t) = 0^T$$

b) Explain why π is know as the stationary distribution X.

ii) A telephone call center receives calls from customers at an average rate of 0.5 per minute. Each cell has a random duration which is exponentially distributed with 3 minutes, independently of the number or duration of any other calls. Two operators are assigned to handle calls. If a call arrives when both operators are busy, the cell is put on hold unless there are already tow cells on hold, in which case the new cell is lost. When a call ends, one of the cells on hold is immediately put through to the newly free operator.

a) Identify the five state which are required if this system is to be modelled as a Markov jump process.
b) Draw the transition diagram for this system.
c) Write down the generator matrix for the process.
d) Evaluate the stationary distribution of the system.

10. Subject CT4 April 2005 Question A6

A life insurance company prices its long-term sickness policies using a three-state Markov model in continuous time. The states are healthy (H), ill (I) and dead (D). The forces of transition in the model are $\sigma_{HI} = \sigma, \sigma_{IH} = p, \sigma_{HD} = \pi, \sigma_{ID} = v$ and they are assumed to be constant overtime.

For a group pf policyholder observed over a 1-year period, there are:

23 transition from State H to State I;

15 transition from State I to State H;

3 deaths from State H;

5 deaths from State I;

The total time spent in state H is 652 years and the total time spent in State I is 44 years.

i) Write down the likelihood function for these data.
ii) Derive the maximum likelihood estimate of σ.
iii) Estimate the standard deviation of δ, the maximum likelihood estimator of σ

11. Subject CT4 September 2005 Question A5

Claims arrive at an insurance company according to a Poisson process with rate λ Per week.

Assume time is expressed in weeks.

i) Show that, given that there is exactly one claim in the time interval [t, t + s], the time of the claim is uniformly distributed on [t, t + s].
ii) State the joint density of the holding times $T_0, T_1, \ldots, T_n$ between successive claims.
iii) Show that, given that there are n claims in the time interval [0, t], the number of claims in the interval [0, s] for s < t is binomial with parameters n and s/t.

12. Subject CT4 September 2005 Question A6

A Markov jump process X_t with state space S = {0, 1, 2,…,N} has the following transition rates;

$$\sigma_{ij} = -\lambda \ \ for\ 0 \leq i \leq N-1$$

$$\sigma_{i,j+1} = \lambda \ \ for\ 0 \leq i \leq N-1$$

$$\sigma_{ij} = 0$$

i) Write down the generator matrix and the Kolmogorov forward equations (in component from) associated with this process.

ii) Verify that for $for\ 0 \leq i \leq N-1$and $j \geq i$, the function

$$Pij(t) = e^{-\lambda} \frac{(\lambda t)^{j-i}}{(j-i)l}$$

Is a solution to the forward equations in (i).

iii) Identify the distribution of the holding times associated with the jump process.

13. Subject CT4 September 2005 Question A1

A time-inhomogeneous Markov jump process has state space {A, B} and the transition rate for switching between states equals 2t , regardless of the state currently occupied, where t is time.

The process starts in state A at t = 0.

i) Calculate the probability that the process remains in state A until at least time 5.

ii) Show that the probability that the process is in state B at time T, and that it is in the first visit to state B, is given by T^2 x exp ($-T^2$).

iii) (a) Sketch the probability function given in (ii).
(b) Give an explanation of the shape of the probability function.
(c) Calculate the time at which it is most likely that the process is in its first visit to state B.

14. Subject CT4 September 2006 Question A2

A time price of a stock can either take a value above a certain point (state A), or take a value below that point (state B). Assume that the evolution of the stock price in time can be modelled by a two-state Markov jump process with homogeneous transition rates $\sigma_{AB} = \sigma, \sigma_{BA} = p$.

The process starts in state A at t = 0 and time is measured in weeks.

i) Write down the generator matrix of the Markov jump process.
ii) State the distribution of the holding time in each of states A and B.
iii) If $\sigma = 3$, find the value of t such that the probability that no transition to state B has occurred until time t is 0.2.
iv) Assuming all the information about the price of the stock is available for a time interval [0, T], explain how the model parameters σ and p can be estimate from the available data.

v) State what you would test to determine whether the data support the assumption of a two-state Markov jump process model for the stock price

15. Subject CT4 September 2006 Question A6

i) Explain the difference between a time-homogeneous and a time inhomogeneous Poisson process.

An insurance company assumes that the arrival of motor insurance claims follow an inhomogeneous Poisson process.

Data on claim arrival times are available for several consecutive years.

ii) a) Describe the main steps in the verification of the company's assumption.
b) State one statistical test that can be used to test the validity of the assumption.

iii) The company concludes that an inhomogeneous Poisson process with rate $\lambda(t) = 3 + \cos(2\pi t)$ is a suitable fit to the claim data (where t is measured in years).

a) Comment on the suitability of this transition rate for motor insurance claims.
b) Write down the Kolmogorov forward equations for $P_{0j}(s, t)$.
c) Verify that these equations are satisfied by:

$$P_{0j}(s,t) = \frac{(f(s,t))^j \exp(-f(s,t))}{j!}$$

For some f(s, t) which you should identify.
[Note that $\int \cos x \, dx = \sin x$.]

d) Comment on the form of the solution compared with the case where λ is constant.

16. Subject CT4 April 2008 Question 11

An investigation was carried out into the relationship between sickness and mortality in an historical population of working class men. The investigation used a three-state model with the states:

1 Heathy
2 Sick
3 Dead

Let the probability that a person in state I at time x will be in state j at time x + t be ${}_tp_x^{ij}$, let the transition intensity at time x + t between any two states I and j ne $p_{x=t}^{ij}$.

i) Draw a diagram showing the three states and the possible transitions between them.
ii) Show from first principles that:

$$\frac{\partial}{\partial t} tP_x^{23} = tP_x^{21}\,\mu_{x+t}^{13} + tP_x^{22}\,\mu_{x+t}^{23}$$

iii) Write down the likelihood of the data in the investigation in terms of the transition rates and the waiting times in the healthy and Stick states, under the assumption that the transition rates are constant.

The investigation collected the following data:

- Man-year in Healthy state 265
- Man-years in Stick state 140
- Number of transitions from Healthy to Stick 20
- Number of transitions from Sick Dead 40

iv) Derive the maximum likelihood of the transition rate from Sick to Dead.
v) Hence estimate:
a) The value of the constant transition rate from Sick to Dead
b) 95 per cent confidence intervals around this transition rate.

17. Subject CT4 September 2008 Question 11

In the village of Selbourne in Southern England in the year 1637 the number of babies born each month was as follows:

January	2	July	5
February	1	August	1
March	1	September	0
April	2	October	2
May	1	November	0
June	2	December	3

Data show that over the 20 years before 1637 there was an average of 1.5 birth per month. You may assume that births in the village historically follow a Poisson process.

An historian has suggested that the large number of births in July 1637 is unusual.

i) Carry out a test of the historian's suggestion, stating your conclusion.

ii) Comment on the assumption that births follow a Poisson process.

18. Subject CT4 September 2009 Question 6

A complaints department of a company has two employees, both of whom work five days per week.

The company models the arrival of complaints using a Poisson process with rate 1.25 per working day.

i) List the assumptions underlying the Poisson process model.
On receipt of a complaint, it is immediately assessed as being straight forward, of medium difficulty or complicated. 60% of cases are assessed as straightforward and 10% are assessed as complicated. The time taken in person-day' effort to prepare responses is assumed to follow an exponential distribution, with parameters 2 for straightforward complaints, 1 for medium difficulty complaints and 0.25 for complicated complaints.

ii) Calculate average number of person-days work expected to be generated by complaints arriving during a five-day working week.

iii) Define a state space under which the number of outstanding complaints can be modelled as a Markov jump process.
The company has a service standard of responding to complaints within a fixed number of days of receipt. It is considering using this Markov jump process to the model the probability of failing to meet this service standard.

iv) Discuss the appropriateness of using the model for this purpose, with reference to the assumption being made

19. Subject CT4 April 2010 Question 11

A reinsurance policy provides cover in respect of a single occurrence of a specified catastrophic event. If such an event occurs, feature cover is suspended. However if a reinstatement premium is paid within one time period of occurrence evet then the insurance coverage is reinstated. If a second specified event occurs it is not permitted to reinstate the cover and the policy will lapse.

The transition rate for the hazard of the specified event is a constant 0.1. whilst policies are eligible for reinstatement, the transition rate for resumption of cover through paying a reinstatement premium is 0.05.

i) Explain whether a time homogeneous or time inhomogeneous model would be more appropriate for modelling this situation.

ii) (a) Explain why a model with state space (cover in force, suspended, lapsed) does not process the Markov property.

(b) Suggest, giving reasons, additional state(s) such that the expanded system would possess the Markov property.

iii) Sketch a transition diagram for the expanded system.

iv) Derive the probability that a policy remains in the cover in force state continuously from time 0 to time t.

v) Derive the probability that a policy is in the suspended state at time $t > 1$ if it is in state cover in force at time 0.

20. Subject CT4 April 2011 Question 9

i) Define a Markov jump process.

A study of a tropical disease used a three-state Markov process model with states:

1. Not suffering from the disease
2. Suffering from the disease
3. Dead.

The disease can be fatal, but most sufferers recover. Let ${}_tp_x^{ij}$ be the probability that a person in state I at age x is in state j at age x + t. Let μ_{x+t}^{ij} be the transition intensity from state I to state j at age x+t.

ii) Show from first principles that:

$$\frac{\partial}{\partial t} {}_tP_x^{13} = {}_tP_x^{11}\,\mu_{x+t}^{13} + {}_tP_x^{12}\,\mu_{x+t}^{23}$$

The study revealed that sufferers who contract the disease a second or subsequent time are more likely to die, and less likely to recover, than first-time sufferers.

iii) Draw a diagram showing the states and possible transitions of a model which allows for this effect yet retains the Markov property.

21. Subject CT4 October 2011 Question 6

A recording instrument is set up to observe a continuous time process, and stores the results for the most recent 250 transitions. The data collected are as follows:

State i	Total time spent in state i (hours)	Number of transition to		
		State A	State B	State C
A	35	Not applicable	60	45
B	150	50	Not applicable	25
C	210	55	15	Not applicable

It is proposed to fit a Markov jump model using the data.

i) (a) State all the parameters of the model.
(b) Outline the assumptions underlying the model.

ii) (a) Estimate the parameters of the model.
(b) Write down the estimated generator matrix of the model.

iii) Specify the distribution of the number of transitions from state i to state j, given the number of transitions out of state i.