INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical & Risk Modelling - 2

Chapter: Unit - 4

Category: Practice Questions

1. Subject CT4 April 2005 Question A3

Let $Y_1, Y_3, Y_5, \ldots$ be a sequence of independent and identically distributed random variables with

$$P(Y_{2k+1} = 1) = P(Y_{2k+1} = -1) = \frac{1}{2}, \quad k = 0,1,2,\ldots$$

And defines $Y_{2k} = Y_{2k+1} = Y_{2k-1}$ for $k = 1,2,3,\ldots$

(i) Show that $\{Y_k : k = 1,2,\ldots\}$ is a sequence of pairwise independent and identically distributed random variables .

Hint : you may use fact that , if X, Y are two variables that take only two values and E(XY) = E(X) E(Y) then X,Y are pairwise independent.

(ii) Explain whether or not $\{Y_k : k = 1,2,\ldots\}$ constitutes a Markov chain.

(iii) (a) State the transition probabilities $p_{ij}(n) = P(Y_{m+n} = j \mid Y_m = i)$ of the sequence $\{Y_k : k = 1,2,\ldots\}$

(b) Hence show that these probabilities do not depend on the current state and that they satisfy the chapman – Kolmogorov equations

2. **Subject CT4 September 2007 Question 9**

In a game of tennis, when the score is at "Deuce" the player winning to next point holds "Advantage". If a player holding "Advantage" wins the following point that player wins the game, but if point is won by the other player yhe score returns to "Deuce".

When Andrew plays tennis against Ben, the probability of Andrews winning any point is 0.6. consider a particular game when the score is at "Deuce"

(i) Show that the subsequent score in the game can be modelled as a Markov Chain, specifying both :
(a) The state space and
(b) The transition matrix.

(ii) State, with reasons, whether the chain is :
(a) Irreducible ; and
(b) Aperiodic.

(iii) Calculate the number of points which must be played before there is more than a 90% chance of the game having been completed.

(iv) (a) Calculate the probability that Andrew wins the game.
(c) Comment on your answer.

3. **Subject CT4 September 2008 Question 8**

A no-Claims Discount system operated by a motor insurer has the following four levels :

Level 1 : 0% discount
Level 2 : 15% discount
Level 3 : 40% discount
Level 4 : 60% discount

The rules for moving between these levels are as follows :

- Following a year with no claims, move to next higher level, of remain at level 4
- Following a year with no claims, move to next higher level, of remain at level 1
- Following a year with two or more claims, move back to levels, or move to level 1 (from level 2) or remain at level 1.

For a given policyholder the probability of no claims in a given year is 0.85 and the probability of making one claim is 0.12.

(i) Write down the transition matrix of this No-Claims Discount process.

(ii) Calculate the probability that a policyholder who is currently at level 2 will be at level 2 after :
(a) One year
(b) Two year

(iii) Calculate the long-run probability that a policyholder is in discount Level 2.

4. Subject CT4 September 2009 Question 7

A firm rents and operates from three locations – the Airport, the Beach and the City. Customers may return vehicles to any of the three locations.

The company estimates that the probability of a car being returned to each location is as follows :

Car returned to

Car hired from	Airport	Beach	City
Airport	0.5	0.25	0.25
Beach	0.25	0.75	0
City	0.25	0.25	0.5

(i) Calculate the 2-steps transition matrix.
(ii) Calculate the stationary distribution π.

It is suggested that the cars should be based at each location in proportion to the stationary distribution.

(iii) Comment on this suggestion.
(iv) Sketch, using your answer to parts (i) and (ii) a graph showing the probability that a car currently located at the Airport is subsequently at the Airport, Beach and City against the number of times the car has been ranted.

5. **Subject CT4 April 2010 Question 10**

An airline runs a frequent flyer scheme with four classes of member: in ascending order Ordinary, Bronze, Silver and Gold. Members receive benefits according to their class. Member who book two or more flight in a given calendar year stay at the same class, and members who book no flights in a given calendar year move down one class (or remain Ordinary members).

Let the proportions of members booking 0, 1 and 2+ flights in a given year be p_0, p_1, and p_{2+} respectively.

(i)

(a) explain how this scheme can be modelled as a Markov chain.
(b) Explain why there must be a unique stationary distribution for the proportion of members in each class.

(ii) Write down the transition matrix of the process.

The airline's research has shown that in any given year, 40% of members book no flights, 40% book exactly one flight, and 20% book two or more flights.

(iii) Calculate the stationary probability distribution.

The cost of running the scheme per member per year is as follows :

- Ordinary members £ 0
- Bronze members £ 10
- Silver members £ 20
- Gold members £ 30

The airline makes a profit of £ 10 per passenger for every flight before taking into account costs associated with the frequently flyer scheme.

(iv) Assess whether the airline makes a profit on the members of the scheme.

6. Subject CT4 October 2010 Question 12

A pet shop has four glass tanks in which snakes for sale are held. The shop can stock at most four snakes at any one time because :

- If more than one snake were held in the same tank, the snakes would attempts to eat each other and
- having snakes loose in the shop would not be popular with the neighbours.

The number of snakes sold by the shop each day is random variable with the following distribution :

Number of Snakes Potentially Sold in Day (if stock is sufficient)	Probability
None	0.4
One	0.4
Two	0.2

If the shop has no snakes in stock at the end of the day, the owner contacts his snake supplier to order four more snakes. The snakes are delivered the following morning before the shop opens. The snake supplier makes a charge of C for the delivery.

(i) Write down the transition matrix for the number of snakes in stock when the shop opens in a morning, given the number in stock when the shop opened the previous day.

(ii) Calculate the stationary distribution for the number of snakes in stock when the shop opens, using your transition matrix in part (i).

(iii) Calculate the expected long term average number of restocking orders placed by the shop owner per trading day.

If a customer arrives intending to purchase a snake, and there is none in stock, the sale is lost to a rival pet shop.

(iv) Calculate the expected long term number of sales lost per trading day.

The owner is unhappy about losing these sales as there is a profit on each sale of P. he therefore consider changing his restocking approach to place an order before he has run out of snakes. The charge for the delivery remains at C irrespective of how many snakes are delivered.

(v) Evaluate the expected number of restocking orders, and number of lost sales per trading day, if the owner decides to restock if there are fewer then two snakes remaining in stock at the end of the day.

(vi) Explain why restocking, when two or more snakes remain in stock cannot optimise the shop's profits.

The pet shop owner wishes to maximize the profit makes on snakes.

(vii) Derive a condition in terms of C and P under which the owner should change from only restocking where there are no snakes in stock, to restocking when there are fewer than two snakes in stock.

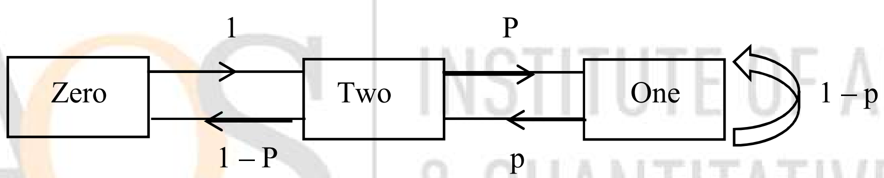
7. **Subject CT4 October 2011 Question 11**

An actuary walks from his house to the office each morning, and walks back again each evening. He owns two umbrellas. If it is raining at the time he sets off, and one or both of his umbrellas is available, he takes an umbrella with him. However if it is raining at the time he sets off he always forgets to take an umbrella.

Assume that the probability of it raining when he sets off on any particular journey is a constant p, independent of other journeys.

This situation is examined as a Markov chain with state space (0, 1, 2) representing the number of his umbrellas at the actuary's current location (office or home) and each time step representing one journey.

(i) Explain why the transition graph for this process is given by:



(ii) Derive the transition matrix for the number of umbrellas at the actuary's house before he leaves each morning, based on the number before he leaves the previous morning.

(iii) Calculate the stationary distribution for the Markov Chain.

(iv) Calculate the long run proportion of journeys (to or from the office) on which the actuary sets out in the rain without an umbrella.

The actuary considers that the weather at the start of a journey, rather than being independent of past history, depends upon the weather at the start of the previous journey. He believes that if it was raining at the start of a journey the probability of it raining at the start at the next journey is r ($0<r<1$), and if it was not raining at the start of a journey the probability of it raining at the start of the journey is s ($0 > s > 1$, $r \neq s$).

(v) Write down the transition matrix for the Markov Chain for the weather.

(vi) Explain why the process with three states (0, 1, 2), being the number of his umbrellas at the actuary's current location, would no longer satisfy this Markov property.

(vii) Describe the additional state (s) needed for the Markov property to be satisfied, and draw a transition diagram for the expanded system.

8. Subject CT4 April 2012 Question 11

The series Y_i records, for each time period i, whether a car driver is accident-free during that period ($Y_j = 0$) or has at least on accident ($Y_i = 1$).

Define $X_i = \sum_{j=1}^{i} Y_j$ with state space (0, 1, 2,. . .).

An insurer makes an assumption about the driver's accident proneness by considering that the probability of a driver having at least one accident is related to the proportion of previous time periods in which the driver has at least one accident as follows:

$$P(Y_{n+1} = 1) = \frac{1}{4}\left(1 + \frac{X_n}{n}\right) \; for\; n \geq 1, with\; P\,(Y_1 = 1) = \frac{1}{2}.$$

(i) Demonstrate that the series X_j satisfies the Markov property, whilst Y_j does not .

(ii) Explain whether the series X_i is :
   (a) Irreducible
   (b) Time – homogeneous.

(iii) Draw the transition graph for X_i covering all transitions which could occur in the three time periods, introducing the transition probabilities.

(iv) Calculate the probability that the driver has accidents during exactly two of the first three time periods.

(v) Comment on the appropriateness of the insurer's assumption about accident proneness

9. Subject CT4 April 2012 Question 12

A company operates a sick pay scheme as follows:

- Healthy employees pay a percentage of salary to found the scheme.
- For the first two consecutive months an employee is sick, the sick pay scheme pays their full salary.
- For the third and subsequent consecutive months of sickness the sick pay is reduced to 50% of full salary.

To simplify administration the scheme operates on whole months only, that is for a particular month's payroll an employee is either healthy or sick for the purpose of the scheme.

The company's experience is that 10% of healthy employee become sick the following month, and that sick employees have a 75% chance of being healthy the next month.

The scheme is to be modelled using a Markov chain.

(i) Explain what is meant by Markov chain.

(ii) Identify the minimum number of states under which the payments under the scheme can be modelled using a time – homogeneous Markov chain. Specifying these states.

(iii) Draw a transition graph for this Markov chain.

(iv) Derive the stationary distribution for this process.

(v) Calculate the minimum percentage of salary which healthy employees should pay for scheme to cover the sick pay costs.

(vi) Calculate the contribution required if, instead, sick pay continued at 100% of salary indefinitely.

(vii) Comment on the benefit to the scheme of the reduction in sick pay to 50% from the third month.

10. Subject 103 April 2004 Question 3

The number, $N(t)$, of members of a pension scheme who are receiving benefits at time t, is subject to change of two kinds:

- It increases by 1 when an active member reaches retirement age
- It decreases by 1 when a retired member dies.

Assume that retirement occur according to a Poisson with rate λ and that each retired member, independently, has a probability μ dt of dying within the time interval $(t, t + dt)$.

i) Explain why, number these assumptions, *N(t)* is a Markov jump process.

ii) Write down the transition rates of N(t).

iii) Using the notation $p_n(t) = P(N(t) = n)$, obtain a differential equation satisfied by $p_n(t)$.

iv) Verify that one solution of the equation (iii) is given by

$$p_n(t) = \frac{1}{n1} e^{-\lambda t \mu} (\frac{\lambda}{\mu})^n, \quad n = 0, 1, \ldots.$$

v) State what conclusions can be drawn from (iv).

11. Subject 103 September 2004 Question 9

Vehicles in a certain country are required to be assessed every year for road-worthiness. At one vehicle assessment center, drivers wait for an average of 15 minutes before the road-worthiness assessment of their vehicle commences. The assessment takes on average 20 minutes to complete. Following assessment, 80% of vehicles are passed as road- worthy allowing the driver to driver home. A further 15% of vehicles are categorised as a “minor fail”; these vehicles require on average 30 minutes of repair work before the driver is allowed to drive home. The remaining 5% of vehicles are categorized as a “significant fail”; these vehicles require on average three hours of repair work before the driver can go home.

A continuous- time Markov model is to be used to model the operation of the vehicle assessment center, with states W (waiting for assessment), A (assessment taking place), M (minor repair taking place), S (significant repair taking place) and H (travelling home).

i) Explain what assumption must be made about the distribution of the time spent in each state.

ii) Write down the generator matrix for this process.

a) Use Kolmogorov’s Forward Equations to write down differential equations satisfied by $p_{WM}(t)$ and by $p_{WA}(t)$.
b) Verify that $p_{WA}(t) = 4e^{-t/20} - 4e^{-t/15}$ for $t \geq 0$, where t is measured in minutes.
c) Derive an expression for $p_{WA}(t)$ for $t \geq 0$

iii) Let T_i be the expected length of time (in minutes) until the vehicle can be driven home given the the assessment process is currently in state i .
a) Explain why $T_W = 15 + T_A$.
b) Derive corresponding equations for T_A, T_M and T_S.
c) Calculate T_W.

12. Subject 104 April 2003 Question 10

An illness-death model has three states:

1 = healthy

2 = sick

3 = dead

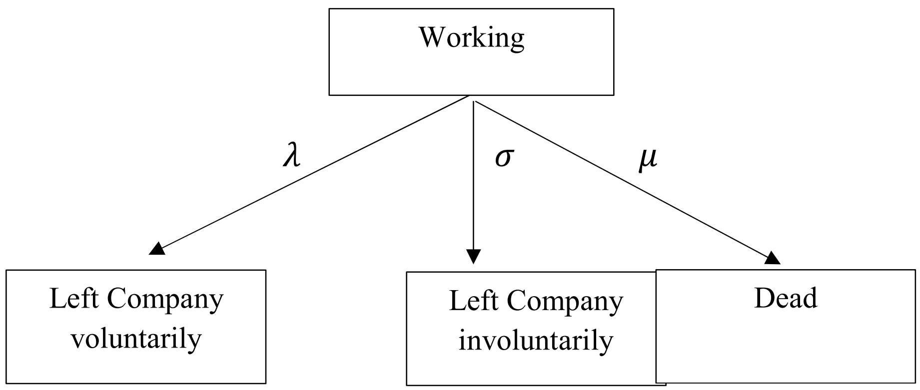
i) Draw and label a diagram showing the three states and the transition intensities between them.

ii) Show, from first principles, that in this illness-death model:

$$\frac{\partial}{\partial t} tP_x^{11} = tP_x^{11}\mu_{x+t}^{12} - tP_x^{12}\mu_{x+t}^{21} - tP_x^{12}\mu_{x+t}^{23}\,.$$

13. Subject 104 April 2003 Question 7 part (i)

A company is modelling its workforce using the multiple state model shown below.



The company is interesting in how many of its employees leave. Employment voluntarily each year and so wishes to estimate the transition intensity λ from the "working" state to the "left company voluntarily" state. Consider employees aged between aged 30 and 31 exact and suppose that the i th individual spends time W_i employed by the company between those ages.

i) Derive the maximum likelihood estimate $\hat{\lambda}$ of λ and write down the variance of $\hat{\lambda}$.

14. Subject CT4 April 2003 Question A4

Marital status is considered using the following time-homogeneous, continuous time Markov jump process:

- The transition rate is from unmarried to married is 0.1 per annum
- The divorce rate is equivalent to a transition rate of 0.05 per annum
- The mortality rate for any individual is equivalent to a transition rate of 0.025 per annum, independent of marital status

The state space of the process consists of five states: Never Married (NM), Married (M), Widowed (W), Divorced (DIV) and Dead (D).

P_x is the probability that a person currently in state x, and who has never previously been windowed, will die without ever being windowed.

i) Construct a transition diagram between the five states.

ii) Show, by general reasoning or otherwise, that P_{NM} equals P_{DIV}.

iii) Demonstrate that:

$$P_{NM} = \frac{1}{5} + \frac{4}{5} \times P_M$$

$$P_M = \frac{1}{4} + \frac{1}{2} \times P_{DIV}$$

iv) Calculate the probability of never being windowed if currently in state NM.

v) Suggest tow ways in which the model could be made more realistic

15. Subject CT4 April 2005 Question A6

An insurance police policy covers the repair of a washing machine, and is subject to a maximum of 3 claims over the year of coverage.

The probability of the machine breaking down has been estimated to follow an exponential distribution with the following annualized frequencies, λ:

$$\lambda \begin{cases} 1/10 & \text{if the machine has not suffered any previous breakdown} \\ 1/5 & \text{if the machine has broken down once previously} \\ 1/4 & \text{if the machine has broken down on two or more occasions} \end{cases}$$

As soon as a breakdown occurs an engineer is dispatched. It can be assumed that repair is made immediately, and that it is always possible to repair the machine.

The washing machine has never broken down at the start of the year (time t = 0).

$P_i(t)$ is the probability that the machine has suffered I breakdowns by time t.

i) Draw a transition diagram for the process defined by the number of breakdowns occurring up to time t.

ii) Write down the Kolmogorov equations obeyed by $P_0(t)$, $P_1(t)$ and $P_2(t)$.

iii) (a) Derive an expression for $P_0(t)$ and

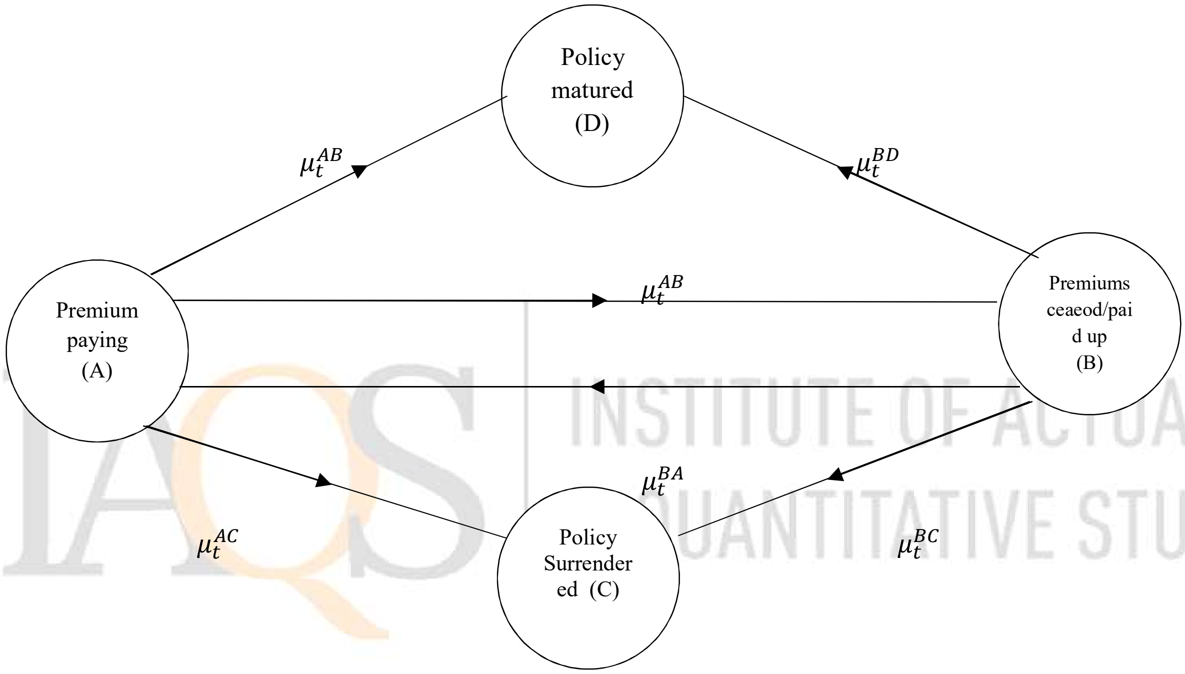
(b) demonstrate that $P_1(t) = e^{-\frac{t}{10}} - e^{-\frac{t}{5}}$.

iv) Derive an expression for $P_2(t)$.

v) Calculate the expected number of claims under the policy.

16. Subject CT4 April 2006 Question A2

A savings provider offers a regular premium pension contract, under which the customer is able to cease paying in premiums and restart them at a later date. In order to profit test, the product, the provider set up four-state Markov model shown in the following diagram:



Show, from first principles, that under this model:

$$\frac{\partial}{\partial t} tP_0^{AA}\, \mu_t^{AB} - tP_0^{AB}(\, \mu_t^{BA} + \mu_t^{BC} + \mu_t^{BD})$$

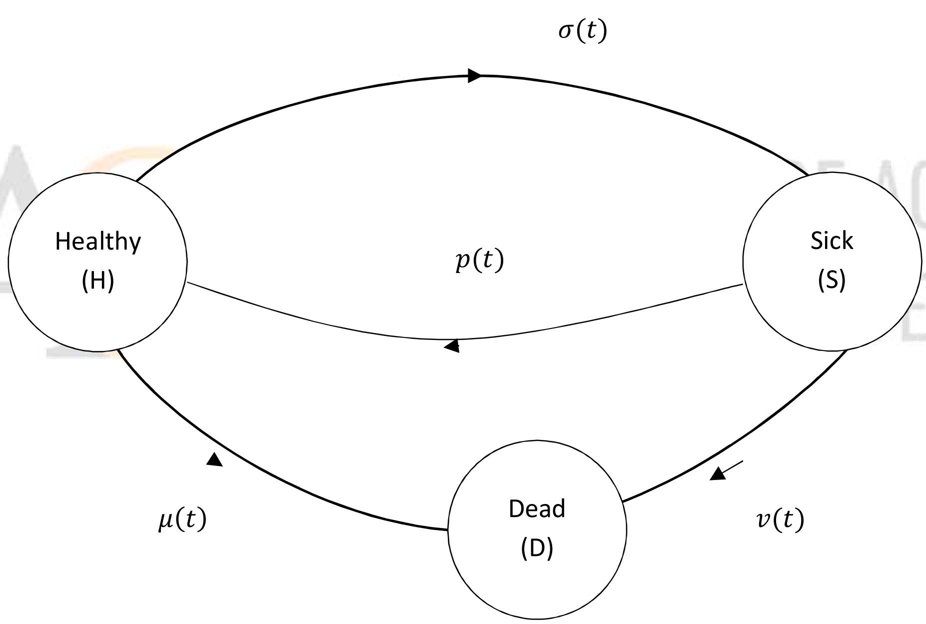
17. Subject CT4 April 2006 Question A6

i) a) Explain what is meant by a Markov jump process.
   b) Explain the condition need for such a process to be time-homogeneous.

ii) Outline the principle difficulties in fitting a Markov jump process model with time-inhomogeneous rates.

A company provides sick pay for a maximum period of six months to its employees who are unable to work.

The following three-state, time-inhomogeneous Markov jump process has been chosen to model future sick pay costs for an individual:



Where "sick" means unable to work and "Healthy" means fit to work.

The time dependence of the transition rates is to reflect increased mortality and morbidity rates as an employee gets older. Time is expressed in years.

iii) Write down Kolmogorov's forward equations for this process, specifying the appropriate transition matrix.

iv) a) Given an employee is sick at time $w < T$, write down an expression for the probability that he or she is sick throughout the period $w < t < T$.

b) Given that a transition out of state H occurred at time w, state the probability that the transition was into state S.

c) For an employee who is healthy at time, τ, given an approximate expression for the probability that there is a transition out of state H in a small-time interval $[w, w + dw]$, where $w > \tau$. Your expression should be in terms of the transition rates and $P_{HH}(\tau, w)$ only.

v) Using the results of part (iv) or otherwise, derive an expression for the probability that an employee is sick at time T and has been sick for less than 6 months, given that they were healthy at time $\tau < T - 1/2$. Your expression should be in terms of the transition rates $P_{HH}(\tau, w)$ only.

vi) Comment on suggestions that:

a) P(t) should also depends on the holding time S, and

b) Mortality rates can be ignored.

18. Subject CT4 April 2007 Question 10 parts (i) and (ii)

The members of particular profession work exclusively in partnerships.

A certain partnership is concerned that it is losing trained technical staff to its competitor, Informal debriefing interviews with individuals leaving the partnership suggest that one reason for this is that the duration elapsing between becoming fully qualified and being made a partner is longer in this partnership than in the profession as a whole.

The partnership decides to investigate whether this claim is true using a multiple- state model with three states: (1) fully qualified but not yet a partner, (2) fully qualified and a partner, (3) working for another partnership. The period of the investigation is to be 1 January 1997 to 31 December 2006.

i) a) Draw and label a state-space diagram depicting the chosen model, showing possible transition between the three states.

b) State any assumption implied by the diagram you have drawn and comment on their appropriateness.

ii) a) State what data be would be required in order to estimate the transition intensity of moving from state (1) to state (2) for employees aged 30 years last birthday.

b) Write down the likelihood of these data.

c) Derive an expression for the maximum likelihood estimate of this transition intensity.

The investigation assumes that all transition intensities are constant within each year of age.

19. Subject CT4 April 2007 Question 11

i) Consider tow Poisson processes, one with rate λ and the other with rate μ. Provide that the sum of events arising from either of these processes is also a Poisson process with rate $(\lambda + \mu)$.

ii) a) Explain what is meant by a Markov jump chain.
b) Describe the circumstances in which the outcome of the Markov jump chain differs from the standard Markov chain with the same transition matrix.

An airline has N adjacent check-in desks at a particular airport, each of which can handle any customer from that airline. Arrivals of passengers at the check-in area are assumed to follow a Poisson process with rate q. The time taken to check-in a passenger is assumed to follow an exponential distribution with1/a.

iii) show the number of desks occupied, together with the number of passengers waiting for a desk to become available, can be formulated as a Markov jump process and specify:

a) the state space; and
b) the transition diagram

iv) State the Kolmogorov forward equations for the process, in component from.

v) Comment on the appropriateness of the assumption made regarding passenger arrival and the check-in process.

vi) a) set out the transition matrix of the jump chain associated with the airline check-in process.
b) Determine the probability that all desks are in use before any passenger has completed the check-in process, given that no passengers have arrived at check-in at the outset.

20. Subject CT4 September 2007 Question 11 (adapted)

The following data have been collected from observation of a three- state process in continuous time;

State occupied	Total time spent in state (hours)	Total transition to;		
		State A	State B	State C
A	50	Not applicable	110	90
B	25	80	Not applicable	45
C	90	120	15	Not applicable

It is proposed to fit a Markov jump model to this data set.

i) a) List all the parameters of the model.
b) Describe the assumptions underlying the model.

ii) a) Estimate the parameters of the model.
b) Give the estimated generation matrix.

The following additional data in respect of secondary transition were collected from observation of the same process.

Triplet of successive transitions	Observed number of triplets n_{ijk}	Triplet of successive transitions	Observed number of triplets n_{ijk}
ABC	42	BCA	38
ABA	68	BCB	7
ACA	85	CAB	64
ACB	4	CAC	56
BAB	50	CBA	8
BAC	30	CBC	7

iii) State the distribution of the number of transitions from state I to state j, given the number of transitions out state i.

iv) Calculate the value of the test statistic for a goodness-of-fit test of the model by considering whither triplet of successive transitions adhere to the distribute give in (iii).

[**Hint**: Use the test statistic:

$$x^2 = \sum_i \sum_j \sum_k \frac{(n_{ijk} - E)^2}{E}$$

Where E is the expected number of triplets under the distribution in (iii).]

v) Identify two other aspects of the appropriateness of the fitted model that could be tested, stating tests in each case.

vi) Outline two methods for simulating the Markov jump process, without performing any calculations.

21. Subject CT4 April 2007 Question 11

An internet service provider. (ISP) is modelling the capacity requirements for its network. It assumes that if a customer is not currently connected to the internet ("offline") the probability of connecting in the short time interval [t, t + dt] is 0.2dt + d(dt). If the customer is disconnecting to the internet ("online") then it assumes the probability of disconnecting in the time interval is given by 0.8dt + o(dt).

The probabilities that the customer is online and offline at time t and $P_{ON}(t)$

And $P_{OFF}(t)$ respectively.

i) Explain why the status of an individual customer can be considered as a Markov Jump Process.

ii) Write down Kolmogorov's forward equation for $P_{OFF}(t)$.

iii) Solve the equation in part (ii) to obtain a formula for the probability that a customer is offline at time t, given that they were offline at 0.

iv) Calculate the expected proportion of time spent online over the period [0, t]. [Hint: Consider the expected value of an indicator function which takes the value 1 if offline and 0 otherwise.]

v) a) Sketch a graph of your answer to (iv) above.
c) Explain its shape.

22. Subject CT4 September 2008 Question 9

A company pension scheme, with a compulsory scheme retirement age of 65, is modelled using a multiple state model with the following categories:

1 Currently employed by the company
2 No longer employed by the company, but not yet receiving a pension
3 Pension in payment, pension commenced early due to ill health retirement
4 Pension in payment, pension commenced at scheme at scheme retirement age
5 Dead

i) Describe the nature of the state space and time space for this process.
ii) Draw and label a transition diagram indicating appropriate transitions between the states.

For i, j in {1, 2, 3, 4, 5}, let:

${}_tP_x^{1j}$ the probability that a life is in state i at age x + t, given they are in state 1 at age x.

μ_{x+1}^{ij} the transition intensity from state i to state j at age x + 1.

iii) Write down equations which could be used to determine the evolution of ${}_tP_x^{23}$ (for each i) appropriate for:

a) $x + t < 65$
b) $x + t = 65$
c) $x + t > 65$

23. Subject CT4 April 2009 Question 8

There is a population of ten cats in a certain neighborhood. Whenever a cat which has fleas meets a cat without fleas, there is a 50% probability that some of the fleas transfer to the other cat such that both cats harbor fleas thereafter, Contacts between two of the neighborhood, cats occur according to a Poisson process with rate μ, and these meetings are equally likely to involve any of the possible pairs of individuals, Assume that once infected a cat continues to have fleas, and that none of the cats' owners has taken any preventative measures.

i) If the number of cats currently infected is x, explain why the number of possible pairings of cats which could result in a new flea infection is
$x(10 - x)$.

ii) Show how the number of infected cats at any time, X(t), can be formulated as a Markov jump process, specifying:

a) The state space

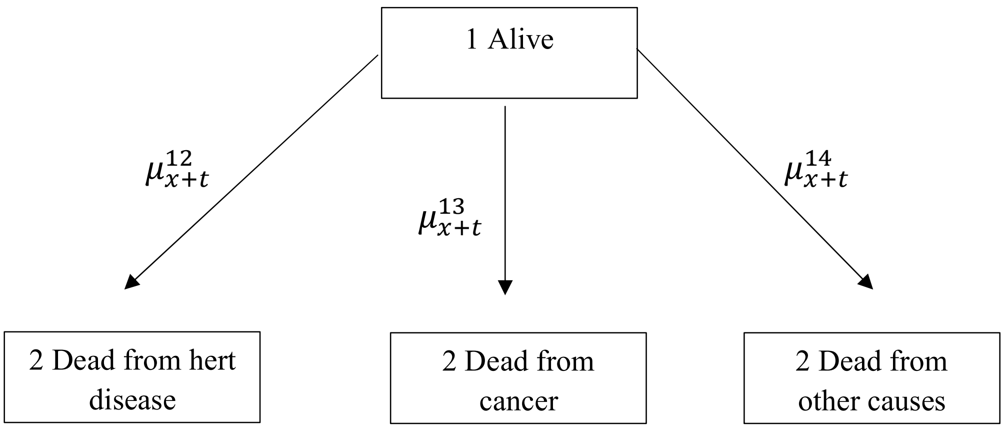
b) The Kolmogorov differential equations in matrix from.

iii) State the distribution of the holding times of the Markov jump process.

iv) Calculate the expected time until all the cats have fleas, starting from a single flea-infected cat.

24. Subject CT4 April 2009 Question 11

An investigation into mortality by cause of death used the four-state Markov model show below.



i) Show from first principles that:

$$\frac{\partial}{\partial t} {}_tP_x^{23} = \mu_{x+t}^{12} \, {}_tP_x^{11}$$

The investigation was carried out separately for each year of age, and the transition intensities were assumed to be constant within each single year of age.

ii) a) write down, defining all terms you use, the likelihood for the transition intensities.
b) Derive the maximum likelihood estimator of the force of mortality from heart disease for any single year of age.
The investigation produced the following data for persons aged 64 last birthday:

Total waiting time in the state Alive	1,065 pension-years
Number of deaths from heart disease	34
Number of deaths from cancer	36
Number of deaths from other causes	42

iii) a) Calculate the maximum likelihood estimate (MLE) of the force of mortality from heart disease at age 64 last birthday.
b) Estimate an approximate 95% confidence interval for the MLS of the mortality from heart disease at age 64 last birthday.

iv) Discuss how you might use this model to analyse the impact of risk factor the on the death rate from heart disease and suggest, giving reasons, a suitable alternative model.

25. Subject CT4 September 2009 Question 8

A researcher is studying a certain incurable disease. The disease can be fate, but often sufferers survive with the condition for a number of years. The research wishes to project the number of number of deaths caused by the disease by using a multiple state model with state space:

{ H – Healthy, I – Infected, $D_{(\text{from disease})}$ ~ Dead (caused by the disease),

$D_{(\text{not from disease})}$ -Dead (not caused by the disease)}.

The transition rates, dependent on age x, are as follows:

- A mortality rate from the Healthy state of $\pi(x)$
- A rate of infection with the disease $\sigma(x)$
- A mortality rate from infected state of v(x) of which p(x) relates to deaths caused by the disease.

i) Draw a transition diagram for the multiple state model.
ii) Write down Kolmogorov's forward equation governing the transitions by specifying the transition matrix.
iii) Determine integral expressions, in terms of the transition rates and any expressions previously determined, for:

a) $P_{HH}(x, x + t)$

b) $P_{HI}(x, x + t)$

c) $P_{HD(\text{from disso})}(x, x +t)$

26. Subject CT4 April 2010 Question 7

A government has introduced a two-tiger driving test system. Once someone applies for a provisional licence they are considered a learner driver. Learner drivers who score 90% or more on the primary examination (which can be taken at any time) become Qualified. Those who score between 50% and 90% are obliged to sit a secondary examination and are given driving status Restricted. Those who score 50% or below on the primary examination remain as Learners. Restricted drivers who pass the secondary examination become Qualified, but those who fail revert back to learner status and obliged to start again.

i) Sketch a diagram showing the possible transitions between the states.

ii) Write down the likelihood of the data, assuming transition rates between states are constant over time, clearly defined all terms you use.

Figure over the first year of the new system based on those who applied for a provisional licence during that time in one area showed the following:

- Person-months in Learner State 1,161
- Person-months in Restricted State 1,940
- Number of transitions from Learner to Restricted 382
- Number of transitions from Restricted to Learner 230
- Number of transitions from Restricted to Qualified 110
- Number of transitions from Learner to Qualified 217

iii) a) Driver the maximum likelihood estimator of the transition rate from Restricted to Learner.

b) Estimate the constant transition rate from Restricted to Learner.

27. Subject CT4 October 2010 Question 10

A study is undertaken of marriage patterns for women in a country where bigamy is not permitted. A sample of women is interviewed and asked about the start and end dates of all their marriages and, where the marriages had ended, whether this was due to death or divorce (all other reasons can be ignored). The investigators are interested in estimating the rate of first marriage for all women and the rate of re-marriage among windows.

i) Draw a diagram illustrating a multiple-state model which the investigators could use to make their estimates, using the four states:
"Never married", "Married", "Widowed" and "Divorced".

ii) Derive from first principles the Kolmogorov differential equation for first marriages.

iii) Write down the likelihood of the data in terms of the waiting times in each state, the numbers of transitions of each type, and the transition intensities, assuming the transition intensities are constant.

iv) Derive the maximum likelihood estimator of the rate of first marriage.

28. Subject CT4 October 2011 Question 8

A continuous-time Markov process with states {Able to work (A), Temporarily unable to work (T), Permanently unable to work (P), dead (D) } is used to model the cost of providing an incapacity benefit when a person is permanently to work.

The generator matrix, with rates expressed per annum, for the process is estimated as:

	A	T	P	D
A	0.15	0.1	0.02	0.03
T	0.45	-0.6	0.1	0.05
P	0	0	-0.2	0.2
D	0	0	0	0

i) Draw the transition graph for the process.

ii) Calculate the probability of a person remaining in state A for at least 5 years continuously.

Define F(i) to be the probability that a person, currently in state i, will never be in state P.

iii) Derive an expression for:
(a) F(A) by conditioning on the first move out of state A.
(b) F(T) by conditioning on the first move out of state T.

iv) Calculate F(A) and F(T).

v) Calculate the expected feature duration spent in state P, for a person currently in state A.

29. Subject CT4 April 2012 Question 10

An investigation was conducted into the effect marriage has on mortality and a model was constructed with three states: 1 Single, 2 Married and 3 Dead. It is assumed that transition rates between states are constant.

i) Sketch a diagram showing the possible transition between states.

ii) Write down an expression for the likelihood of the data in terms of transition rates and waiting times, defining all the terms you use.

The following data were collected from information on males and females in their thirties.

Years spent in Married state	40,062
Years spent in Single state	10,298
Number of transitions from Married to Single	1,382
Number of transitions from Single to Dead	12
Number of transitions from Married to Dead	9

iii) Derive the maximum likelihood estimator of the transition rate from single to Dead.

iv) Estimate the constant transition rate from Single to Dead and its variance.

30. Subject CT4 September 2012 Question 7

The volatility of equity is classified as being High (H) or Low (L) according to whether it is above or below a particular level. The volatility status is assumed to follow a Markov jump process with constant transition rates $\emptyset_{LH} = \mu \; and \; \emptyset_{LH} = p.$

i) Write down the generator matrix of the Matrix jump process.

ii) State the distribution of holding times in each state.

A history of equity price volatility is available over a representative time period

iii) Explain how the parameter μ and p can be estimated.

Let ${}_tP_s^{ij}$ be the probability that the process is in state j at time s + t given that it was in state i at times s (I, j = H, L), where t $\geq$ 0. Let ${}_tP_s^{\overline{ij}}$ be the probability that the process remains in state I from time s to s + t.

iv) Write down Kolmogorov's forward equation for $\frac{\partial}{\partial t}\,{}_tP_s^{\overline{LL}}$, $\frac{\partial}{\partial t}\,{}_tP_s^{LL}$ and $\frac{\partial}{\partial t}\,{}_tP_s^{LH}$.

Equity price volatility is low at time zero.

v) Derive an expression for the time after which there is a greater than 50% chance of having experienced a period of high equity price volatility.

vi) Solve the Kolmogorov equation to obtain an expression for ${}_tP_s^{LL}$.

31. Subject CT4 September 2012 Question 10

On a small distant planet lives a race of aliens. The aliens can die in one of two ways, either through illness, or by being sacrificed according to the ancient custom of the planet. Aliens who die from either cause may, sometime later, become zombies.

i) Draw a multiple-state diagram with four states illustrating the process by which aliens die and become zombies, labelling the four states and the possible transitions between them.

ii) Write down the likelihood of the process in terms of the transition intensities, the numbers of events observed and the waiting times in the relevant states, clearly defining all the terms you use.

iii) Derive the maximum likelihood estimator of the death rate from illness.

The aliens take censuses of their population every ten years (where the year is an "alien year", which is the length of time planet takes to orbit their sun). On 1 January in alien year 46,567, there were 3,189 live aliens in the population. On 1 January in alien year 46,577, there were 2,811 live aliens in the population. During the intervening ten alien years, a total of 3,690 aliens died from illness and 2,310 were sacrificed, and the annual death rates from illness and sacrifice were constant and the same for each alien.

iv) Estimate the annual death rates from illness and from sacrifice over the ten alien years between alien years 46,567 and 46,577.
The rate at which aliens who have died from either cause become zombies is 0.1 per alien year.

v) Calculate the probabilities that an alive in alien year 46,587 will, ten alien years later:
(a) Still be alive
(b) Be dead but not a zombie.

32. Subject CT4 April 2013 Question 8

During a football match, the referee can caution players if they commit an offence by showing them a yellow card. If a player commits a second offence which the referee deems worthy of a caution, rate are show a red card, and are sent off the pitch and take no further part in the match. If the referee considers a particularly serious offence has been committed, he can show a red to a player who has not previously been cautioned, and send the player off immediately.

The football team manager can also decide to substitute one player for another at any point in the match so that the substitute player take no further part in the match. Due to risk of a player being sent off, the manager is more likely to substitute a player who has been shown a yellow card. Experience shows that players who have been shown a yellow card play more carefully to try to avoid a second offence.

The rate at which uncautioned players are show a yellow card is 1/10 per hour.

The rate at which those players who have already been shown a yellow card are shown a red card is 1/15 per hour.

The rate at which uncautioned players are shown a red card is 1/40 per hour.

The rate at which players are substituted is 1/10 per hour if they have not been shown a yellow card, and 1/15 if they have been shown a yellow card.

i) Sketch a transition graph showing the possible transitions between states for a given player.

ii) Write down the compact from of the Kolmogorov forward equations, specifying the generator matrix.

A football match lasts 1.5 hours.

iii) Solve the Kolmogorov equation for the probability that a player who starts the match remains in the game for the whole match without being show a yellow card or a red card.

iv) Calculate the probability that player who starts the match is sent off during the match without previously having been cautioned.

Consider a match that continued indefinitely rather than ending after 1.5 hours.

v) (a) Derive the probability that in this instance a player is sent off without previously having been cautioned.

(c) Explain your result.

33. Subject CT4 April 2013 Question 10

i) State the Markov property.

A certain non-fatal medical condition affects adults. Adults with the condition suffer frequent episodes of blurred vision. A study was carried out among a group of adults known to have the condition. The study lasted one year, and each participant in the study was asked to record the duration of each episode of blurred vision. All participants remained under observation for the entire year.

The data from the study were analysed using a two-state model with states:

1. Not suffering from blurred vision
2. Suffering from blurred vision.

Let the transition rate from state i to state j at time x + t be μ_{x+t}^{ij} , and let the probability that a person in state I at time x will be in state j at time x + t P_s^{ij} .

ii) Derive from first principle the Kolmogorov forward equation for the transition from state 1 to state 2.

The results of the study were as follows:

Participant-days in state 1	21,650
Participant-days in state 2	5,200
Number of transitions from state 1 to state 2	4,330
Number of transitions from state 2 to state 1	4,160

Assume of transition intensities constant over time.

iii) Calculate the maximum likelihood estimates of the transition intensities from state 1 to state 2 and from state 2 to state 1.

iv) Estimate the probability that an adult with the condition who is presently not suffering from blurred vision will be suffering from blurred vision in 3 days' time.