Lecture

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Chapter Name: Portfolio Risk: Analytical methods

1 Portfolio

1.1 What is portfolio?

- A portfolio can be characterized by positions on a certain number of constituent assets, expressed in the base currency, say, dollars.
- If the positions are fixed over the selected horizon, the portfolio rate of return is a linear combination of the returns on underlying assets, where the weights are given by the relative amounts invested at the beginning of the period.
- Define the portfolio rate of return from t to $t + 1$ as:

$$\boldsymbol{R_{p,t+1} = \sum_{i=1}^{N} w_i R_{i,t+1}}$$

- where N is the number of assets, $R_{i,t+1}$ is the rate of return on asset i, and w_i is the weight. The rate of return is defined as the change in the dollar value, or dollar return, scaled by the initial investment. This is a unitless measure.

1 Portfolio

- Weights are constructed to sum to unity by scaling the dollar positions in each asset W_i by the portfolio total market value W. This immediately rules out portfolios that have zero net investment W = 0, such as some derivatives positions.
- But we could have positive and negative weights w_i, including values much larger than 1, as with a highly leveraged hedge fund. If the net portfolio value is zero, we could use another measure, such as the sum of the gross positions or absolute value of all dollar positions W^*.
- All weights then would be defined in relation to this benchmark. Alternatively, we could express returns in dollar terms, defining a dollar amount invested in asset i as $W_i = w_i W$. We will be using x as representing the vector of dollar amount invested in each asset so as to avoid confusion with the total dollar amount W.
- It is important to note that in traditional mean-variance analysis, each constituent asset is a security. In contrast, VAR defines the component as a risk factor and w_i as the linear exposure to this risk factor.

1 Diversified Portfolio VAR

- Portfolio theory depends a lot on statistical assumptions. In finance, researchers and analysts often assume returns are normally distributed. Such an assumption allows us to express relationships in concise expressions such as beta.
- Actually, beta and other convenient concepts can apply if returns follow an elliptical distribution, which is a broader class of distributions that includes the normal distribution. In what follows, we will assume returns follow an elliptical distribution unless otherwise stated.
- Diversified VaR is simply the VaR of the portfolio where the calculation takes into account the diversification effects. The basic formula is:

$$\boldsymbol{VaR_p = Z_c * \sigma_P * P}$$

- where:
- Z_c = the z-score associated with the level of confidence c
- σ_P = the standard deviation of the portfolio return
- P = the nominal value invested in the portfolio

1 Diversified Portfolio VAR

- Examining the formula for the variance of the portfolio returns is important because it reveals how the correlations of the returns of the assets in the portfolio affect volatility. The variance formula is:

$$\sigma_P^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j$$

- where:
- σ_P^2 = the variance of the portfolio returns
- w_i = the portfolio weight invested in position i
- σ_i = the standard deviation of the return in position i
- $\rho_{i,j}$ = the correlation between the returns of asset i and asset j

- The standard deviation, denoted σ_P, is:

$$\sigma_P = \sqrt{\sigma_P^2} = \sqrt{\sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_i w_j \rho_{i,j} \sigma_i \sigma_j}$$

- Clearly, the variance and standard deviation are lower when the correlations are lower.

1 Individual VAR

- Individual VaR is the VaR of an individual position in isolation. If the proportion or weight in the position is w_i, then we can define the individual VaR as:

$$VaR_i = Z_c \times \sigma_i \times |P_i| = Z_c \times \sigma_i \times |w_i| \times P$$

- where:
- P = the portfolio value
- P_i = the nominal amount invested in position I

- We use the absolute value of the weight because both long and short positions pose risk.

1 Portfolio VAR

- In order to calculate delta-normal VaR with more than one risk factor, we need a covariance matrix that incorporates correlations between each risk factor in the portfolio and volatilities of each risk factor.
- If we know the volatilities and correlations, we can derive the standard deviation of the portfolio and the corresponding VaR measure. We will discuss how to calculate VaR using matrix multiplication next.

1 Portfolio VAR

- The portfolio return can be written using matrix notation, replacing a string of numbers by a single vector:

$$R_p = w_1 R_1 + w_2 R_2 + \cdots + w_N R_N = [w_1 w_2 \cdots w_N] \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_N \end{bmatrix} = w' R$$

- where w' represents the transposed vector (i.e., horizontal) of weights, and R is the vertical vector containing individual asset returns.
- The portfolio expected return is $$E(R_p) = \mu_p = \sum_{i=1}^{N} w_i \mu_i$$

- and the variance is $$V(R_p) = \sigma_p^2 = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + \sum_{i=1}^{N} \sum_{j=1, j \neq i}^{N} w_i w_j \sigma_{ij} = \sum_{i=1}^{N} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{N} \sum_{j<i}^{N} w_i w_j \sigma_{ij}$$

- This sum accounts not only for the risk of the individual securities σ_i^2 but also for all covariances, which add up to a total of N(N-1)/2 different terms.

1 Portfolio VAR

- As the number of assets increases, it becomes difficult to keep track of all covariance terms, which is why it is more convenient to use matrix notation. The variance can be written as

$$\sigma_p^2 = [w_1 \cdots w_N] \begin{bmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \ldots & \sigma_{1N} \\ \vdots & & & & \\ \sigma_{N1} & \sigma_{N2} & \sigma_{N3} & \ldots & \sigma_N^2 \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}$$

- Defining $\sum$ as the covariance matrix, the variance of the portfolio rate of return can be written more compactly as

$$\sigma_p^2 = w' \Sigma w$$

- where *w* are weights, which have no units. This also can be written in terms of dollar exposures *x* as

$$\sigma_p^2 W^2 = x' \Sigma x$$

1 Portfolio VAR

- So far nothing has been said about the distribution of the portfolio return. Ultimately, we would like to translate the portfolio variance into a VAR measure. To do so, we need to know the distribution of the portfolio return.
- In the delta-normal model, all individual security returns are assumed normally distributed. This is particularly convenient because the portfolio return, a linear combination of jointly normal random variables
- is also normally distributed.
- If so, we can translate the confidence level c into a standard normal deviate α such that the probability of observing a loss worse than $-\alpha$ is c. Defining W as the initial portfolio value, the portfolio VAR is

$$\text{Portfolio VAR} = \text{VAR}_p = \alpha\sigma_p W = \alpha\sqrt{x'\Sigma x}$$

1 Diversified VAR

- Diversified VaR is simply the VaR of the portfolio where the calculation takes into account the diversification effects.
- The portfolio VAR, taking into account diversification benefits between components.

1 Diversified VAR

- The basic formula according to elliptical distribution Diversified VAR is:

$$VaR_p = Z_c \times \sigma_p \times P$$

- where:
- Z_c = the z-score associated with the level of confidence c
- σ_P = the standard deviation of the portfolio return
- P= the nominal value invested in the portfolio

- Examining the formula for the variance of the portfolio returns is important because it reveals how the correlations of the returns of the assets in the portfolio affect volatility. The variance formula is:

$$\sigma_P{}^2 = \sum_{i=1}^{N} w_i{}^2\sigma_i{}^2 + 2\sum_{i=1}^{N}\sum_{j<i}^{N} w_i w_j \rho_{i,j}\sigma_i\sigma_j$$

- where:
- σ_P^2 = the variance of the portfolio returns
- w_i = the portfolio weight invested in position i
- σ_i = the standard deviation of the return in position i
- $\rho_{i,j}$ = the correlation between the returns of asset i and asset j

1 Marginal VAR

- Marginal VaR applies to a particular position in a portfolio, and it is the per unit change in a portfolio VaR that occurs from an additional investment in that position.
- To measure the effect of changing positions on portfolio risk, individual VARs are not sufficient. Volatility measures the uncertainty in the return of an asset, taken in isolation.
- When this asset belongs to a portfolio, however, what matters is the contribution to portfolio risk.

- According to elliptical distribution, it is the partial derivative of the portfolio VaR with respect to the position:

$$\text{Marginal VaR} = \text{MVaR}_i = \frac{\partial \text{VaR}_P}{\partial(\text{monetary investment in i})} = Z_c \frac{\partial \sigma_P}{\partial w_i} = Z_c \frac{\text{cov}(R_i, R_P)}{\sigma_P}$$

1 Marginal VAR

- Using CAPM methodology, we know a regression of the returns of a single asset i in a portfolio on the returns of the entire portfolio gives a beta, denoted β_i which is a concise measure that includes the covariance of the position's returns with the total portfolio:

$$\beta_i = \frac{\mathrm{cov}(R_i, R_P)}{\sigma_P{}^2}$$

- Using the concept of beta gives another expression for marginal VaR:

$$\text{Marginal VaR} = \text{MVaR}_i = \frac{\text{VaR}_P}{\text{portfolio value}} \times \beta_i$$

1 Marginal VAR

- The delta-normal distribution marginal VAR is closely related to the beta, defined as

$$\beta_i = \frac{\operatorname{cov}(R_i, R_p)}{\sigma_p^2} = \frac{\sigma_{ip}}{\sigma_p^2} = \frac{\rho_{ip}\sigma_i\sigma_p}{\sigma_p^2} = \rho_{ip}\frac{\sigma_i}{\sigma_p}$$

- Beta is also called the systematic risk of security i vis-à-vis portfolio p and can be measured from the slope coefficient in a regression of R_i on R_P.
- Beta risk is the basis for capital asset pricing model (CAPM) developed by Sharpe (1964). According to the CAPM, well-diversified investors only need to be compensated for the systematic risk of securities relative to the market. In other words, the risk premium on all assets should depend on beta only.

- To summarize, the relationship between the Δ(Marginal)VAR and Β is:

$$\Delta\text{VAR}_i = \frac{\partial\text{VAR}}{\partial x_i} = \alpha(\beta_i \times \sigma_p) = \frac{\text{VAR}}{W} \times \beta_i$$

1 Incremental VAR

- Incremental VAR is the change in VAR owing to a new position. It differs from the marginal VAR in that the amount added or subtracted can be large, in which case VAR changes in a nonlinear fashion.

- Incremental VaR is the change in VaR from the addition of a new position in a portfolio. Since it applies to an entire position, it is generally larger than marginal VaR and may include nonlinear relationships, which marginal VaR generally assumes away.

- The problem with measuring incremental VaR is that, in order to be accurate, a full revaluation of the portfolio after the addition of the new position would be necessary.

- The incremental VaR is the difference between the new VaR from the revaluation minus the VaR before the addition. The revaluation requires not only measuring the risk of the position itself but it also requires measuring the change in the risk of the other positions that are already in the portfolio.

- For a portfolio with hundreds or thousands of positions, this would be time consuming. Clearly, VaR measurement becomes more difficult as portfolio size increases given the expansion of the covariance matrix. Using a shortcut approach for computing incremental VaR would be beneficial.

1 Incremental VAR

- Following elliptical distribution, for small additions to a portfolio, we can approximate the incremental VaR with the following steps:
  - Step 1: Estimate the risk factors of the new position and include them in a vector [η] .
  - Step 2: For the portfolio, estimate the vector of marginal VaRs for the risk factors [MVaR_j] .
  - Step 3: Take the cross product.

- This probably requires less work and is faster to implement because it is likely the managers already have estimates of the vector of [MVaR_j] values in Step 2.

- For delt-normal distributions, this methodology can be extended to evaluate the total impact of a proposed trade on portfolio p. A new trade is represented by position a, which is a vector of additional exposures to our risk factors, measured in dollars.
- Ideally, we should measure the portfolio VAR at the initial position VAR_p and then again at the new position VAR_{p+a}. The incremental VAR then is obtained as:

$$\text{Incremental VAR} = \text{VAR}_{p+a} - \text{VAR}_p$$

- The incremental VAR can be reported as, approximately as:

$$\text{Incremental VAR} \approx (\Delta\text{VAR})' \times a$$

1 Portfolio management using VAR

- A manager can lower a portfolio VaR by lowering allocations to the positions with the highest marginal VaR. If the manager keeps the total invested capital constant, this would mean increasing allocations to positions with lower marginal VaR. Portfolio risk will be at a global minimum where all the marginal VaRs are equal for all *i* and *j*:

$$\boldsymbol{MVaR_i = MVaR_j}$$

- The next step is to consider the portfolio expected return as well as its risk. Indeed, the role of the portfolio manager is to choose a portfolio that represents the best combination of expected return and risk. Thus we are moving from risk management to portfolio management.
- Risk management focuses on risk and ways to reduce risk; however, minimizing risk may not produce the optimal portfolio. Portfolio management requires assessing both risk measures and return measures to choose the optimal portfolio.
- Traditional efficient frontier analysis tells us that the minimum variance portfolio is not optimal. We should note that the efficient frontier is the plot of portfolios that have the lowest standard deviation for each expected return (or highest return for each standard deviation) when plotted on a plane with the vertical axis measuring return and the horizontal axis measuring the standard deviation.

1 Portfolio management using VAR

- The optimal portfolio is represented by the point where a ray from the risk-free rate is just tangent to the efficient frontier. That optimal portfolio has the highest Sharpe ratio:

$$\boldsymbol{Sharpe\ Ratio = \frac{portfolio\ return\ - risk\ free\ rate}{standard\ deviation\ of\ portfolio\ return}}$$

- We can modify this formula by replacing the standard deviation with VaR so that the focus then becomes the excess return of the portfolio over VaR:

$$\boldsymbol{\frac{portfolio\ return - risk\ free\ rate}{VaR\ of\ portfolio}}$$

- This ratio is maximized when the excess return in each position divided by its respective marginal VaR equals a constant. In other words, at the optimum:

$$\boldsymbol{\frac{Position\ i\ return\ - risk\ free\ rate}{MVaR_i} = \frac{Position\ j\ return\ - risk\ free\ rate}{MVaR_j}}$$

- for all positions i and j.

1 Portfolio management using VAR

- Assuming that the returns follow elliptical distributions, we can represent the condition in a more concise fashion by employing betas, β_i which are obtained from regressing each position's return on the portfolio return:

$$\frac{\boldsymbol{Position\ i\ return\ - risk\ free\ rate}}{\boldsymbol{\beta_i}} = \frac{\boldsymbol{Position\ j\ return\ - risk\ free\ rate}}{\boldsymbol{\beta_j}}$$

- for all positions i and j.

- The portfolio weights that make these ratios equal will be the optimal portfolio.

1 Portfolio management using VAR