Revision Lecture

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M Sc. Sem 1

Subject : Probability and Statistics

Revision and Questions

1.1 Question?

Calculate the probability that at least 9 out of a group of 10 people who have been infected by a serious disease will survive, if the survival probability for the disease is 70%.

Solution

The number of survivors is distributed binomially with parameters $n=10$, and $p=0.7$. If X is the number of survivors, then:

$$P(X \geq 9) = P(X = 9 \text{ or } 10) = \binom{10}{9} \times 0.7^9 \times 0.3 + \binom{10}{10} \times 0.7^{10} = 0.1493$$

1.1 Question?

If the probability of having a male or female child is equal, calculate the probability that a woman's fourth child is her first son.

Solution

The probability is $\left(\frac{1}{2}\right)^3 \times \frac{1}{2} = 0.0625$.

1.1 Question?

If goals are scored randomly in a game of football at a constant rate of three per match, calculate the probability that more than 5 goals are scored in a match.

Solution

The number of goals in a match can be modelled as a Poisson distribution with mean $\lambda = 3$.

$$P(X > 5) = 1 - P(X \leq 5)$$

We can use the recurrence relationship given:

$$P(X = 0) = e^{-3} = 0.0498$$

$$P(X = 1) = \frac{3}{1} \times 0.0498 = 0.1494$$

$$P(X = 2) = \frac{3}{2} \times 0.1494 = 0.2240$$

$$P(X = 3) = \frac{3}{3} \times 0.2240 = 0.2240$$

$$P(X = 4) = \frac{3}{4} \times 0.2240 = 0.1680$$

$$P(X = 5) = \frac{3}{5} \times 0.1680 = 0.1008$$

So we have $P(X > 5) = 1 - 0.9161 = 0.0839$.

1.1 Question?

If each of the 55 million people in the UK independently has probability □ □ 8 1 10 of being killed by a falling meteorite in a given year, use an approximation to calculate the probability of exactly 2 such deaths occurring in a given year.

Solution

If X is the number of people killed by a meteorite in a year then X has a binomial distribution with $n = 55{,}000{,}000$ and $p = 1 \times 10^{-8}$. We can approximate this by using a Poisson distribution with:

$$\lambda = np = 55{,}000{,}000 \times 1 \times 10^{-8} = 0.55$$

Hence:

$$P(X = 2) = \frac{0.55^2}{2!} e^{-0.55} = 0.0873$$

1.1 Question?

The number of home insurance claims a company receives in a month is distributed as a Poisson random variable with mean 2. Calculate the probability that the company receives exactly 30 claims in a year. Treat all months as if they are of equal length.

Solution

Let X denote the number of home insurance claims received in a year. Since the number of claims in a month has a $Poi(2)$ distribution; $X \sim Poi(24)$. The required probability is:

$$P(X=30)=\frac{24^{30}}{30!}e^{-24}=0.0363$$

Alternatively, we could use the cumulative Poisson probabilities given on page 184 of the Tables:

$$P(X=30)=P(X\leq 30)-P(X\leq 29)=0.90415-0.86788=0.0363$$

1.1 Question?

?

Determine the median of the $Exp(\lambda)$ distribution. (The median is the value of m such that $P(X \leq m) = \frac{1}{2}$.)

Solution

Since $P(X \leq m) = F_X(m)$, we have:

$$1 - e^{-\lambda m} = 0.5 \Rightarrow 0.5 = e^{-\lambda m} \Rightarrow -\lambda m = \ln 0.5 \Rightarrow m = -\frac{1}{\lambda}\ln 0.5$$

Since $\ln 0.5 = -\ln 2$, we can say $m = \frac{\ln 2}{\lambda}$.

1.1 Question?

If the random variable X has a χ_5^2 distribution, calculate:

(a) $P(X > 6.5)$

(b) $P(X < 11.8)$.

Solution

Using the χ^2 probabilities given on pages 164–166 of the *Tables*, we obtain:

(a) $1 - 0.7394 = 0.2606$

(b) Here we need to interpolate between the two closest probabilities given, *ie* $P(X < 11.5) = 0.9577$ and $P(X < 12) = 0.9652$, so:

$$P(X < 11.8) \approx 0.9577 + \frac{11.8 - 11.5}{12 - 11.5} \times (0.9652 - 0.9577) = 0.9622$$

Alternatively, we could use interpolation on the χ^2 percentage points tables given on page 168-169 of the Tables. These give the approximate answers of 0.2644 and 0.9604.

1.1 Question?

If reported claims follow a Poisson process with rate 5 per day (and the insurer has a 24 hour hotline), calculate the probability that:

(i) there will be fewer than 2 claims reported on a given day

(ii) the time until the next reported claim is less than an hour.

Solution

(i) The number of claims per day, X, has a $Poi(5)$ distribution, so:

$$P(X<2)=P(X=0)+P(X=1)$$
$$=e^{-5}+5e^{-5}$$
$$=0.0404$$

Alternatively, we can read the value of $P(X\le 1)$ from the cumulative Poisson tables listed on page 176 of the Tables.

(ii) The number of claims per hour, Y, has a $Poi\left(\frac{5}{24}\right)$ distribution, so the waiting time (in hours), T, has an $Exp\left(\frac{5}{24}\right)$ distribution. Hence:

$$P(T<1)=\int_0^1 \frac{5}{24}e^{-\frac{5}{24}t}\,dt=\left[-e^{-\frac{5}{24}t}\right]_0^1=1-e^{-\frac{5}{24}}=0.188$$

1.1 Question?

1.11 Claim amounts are modelled as an exponential random variable with mean £1,000.

xam style

(i) Calculate the probability that a randomly selected claim amount is greater than £5,000. [1]

(ii) Calculate the probability that a randomly selected claim amount is greater than £5,000 given that it is greater than £1,000. [2]

[Total 3]

1.11 (i) ***Probability***

$$P(X>5{,}000)=1-F(5{,}000)=e^{-0.001\times 5{,}000}=e^{-5}$$

$$=0.00674$$ [1]

(ii) ***Conditional probability***

$$P(X>5{,}000\,|\,X>1{,}000)=\frac{P(X>5{,}000\cap X>1{,}000)}{P(X>1{,}000)}$$

$$=\frac{P(X>5{,}000)}{P(X>1{,}000)}$$ [1]

We have already found the numerator, we just need to find the denominator:

$$P(X>1{,}000)=1-F(1{,}000)=e^{-0.001\times 1{,}000}=e^{-1}$$

So the required probability is:

$$P(X>5{,}000\,|\,X>1{,}000)=\frac{e^{-5}}{e^{-1}}=e^{-4}=0.0183$$ [1]

1.1 Question?

1.14 It is assumed that claims arising on an industrial policy can be modelled as a Poisson process at a rate of 0.5 per year.

am style

(i) Determine the probability that no claims arise in a single year. [1]

(ii) Determine the probability that, in three consecutive years, there is one or more claims in one of the years and no claims in each of the other two years. [2]

(iii) Suppose a claim has just occurred. Determine the probability that more than two years will elapse before the next claim occurs. [2]

[Total 5]

(i) ***Probability of no claims in one year***

The distribution of the number of claims, N, in one year is $Poi(0.5)$. Hence the probability of no claims in one year is:

$$P(N=0)=\frac{0.5^0}{0!}e^{-0.5}=0.60653 \qquad [1]$$

(ii) ***Probability of no claims in two of three years***

Using our result from part (i), the probability of one or more claims in one year is:

$$P(N\geq 1)=1-P(N=0)=1-0.60653=0.39347$$

If X is the number of years with one or more claim, then:

$$X \sim Bin(3,0.39347)$$

So we have:

$$P(X=1)={}^3C_1\times 0.39347\times 0.60653^2=0.43425 \qquad [2]$$

(iii) ***Probability that more than two years will elapse before the next claim***

The waiting time, T, in years has an $Exp(0.5)$ distribution.

$$P(T>2)=1-F(2)=e^{-0.5\times 2}=e^{-1}=0.36788 \qquad [2]$$