

Class: TY BSc

Subject : Statistical and Risk Modelling - 3

Chapter: Unit 4 Chapter 1 (Part 2)

Chapter Name: Markov process (Time-inhomogeneous)

Today's Agenda

1. Time-inhomogeneous Markov Jump process
2. Chapman-Kolmogorov Equations
3. Transition Rates
4. Kolmogorov's Differential Equations
 1. Forward
 2. Backward
5. Probability
6. Integrated form of the Kolmogorov equations
 1. Backward
 2. Forward

1 Time inhomogeneous Markov Jump Process



Definition - Time-inhomogeneous Markov jump process are processes in which the transition probabilities $P(X_t = j \mid X_s = i)$ depend not only on the length of the time interval $[s, t]$, but also on the times s and t when it starts and ends.

This is because the transition rates for a time-inhomogeneous process vary over time.

2 The Chapman-Kolmogorov equations

The more general continuous-time Markov jump process $\{X_t, t \geq 0\}$ has transition probabilities:

$$p_{ij}(s, t) = P[X_t = j \mid X_s = i] \quad (s \leq t)$$

which obey a version of the Chapman-Kolmogorov equations, written in matrix form as:

$$P(s, t) = P(s, u)P(u, t) \quad \text{for all } s < u < t$$

or equivalently:

$$p_{ij}(s, t) = \sum_{k \in S} p_{ik}(s, u) p_{kj}(u, t) \quad \text{for all } s < u < t$$

3 Transition Rates

Proceeding as in the time-homogeneous case, we obtain:

$$p_{ij}(s, s + h) = \begin{cases} h\mu_{ij}(s) + o(h) & \text{if } i \neq j \\ 1 + h\mu_{ii}(s) + o(h) & \text{if } i = j \end{cases}$$

We see that the only difference between this case and the time-homogeneous case studied earlier is that the transition rates $\mu_{ij}(s)$ are allowed to change over time.

4.1 Kolmogorov's forward differential equations



Kolmogorov's forward differential equations (time-inhomogeneous case) These can be written in compact (i.e., matrix) form as:

$$\frac{\partial}{\partial t} P(s, t) = P(s, t) A(t)$$

where $A(t)$ is the matrix with entries $\mu_{ij}(t)$.

Occupancy Probabilities

For a time-inhomogeneous Markov jump process:

$$p_{\bar{i}\bar{i}}(s, t) = \exp\left(-\int_s^t \lambda_i(u) du\right) = \exp\left(-\int_0^{t-s} \lambda_i(s+u) du\right)$$

where $\lambda_i(u)$ denotes the total force of transition out of state i at time u .

4.2 Kolmogorov's backward differential equations



Kolmogorov's backward differential equations (time-inhomogeneous case)

The matrix form of Kolmogorov's backward equations is:

$$\frac{\partial}{\partial s} P(s, t) = -A(s)P(s, t)$$

It is still the case that:

$$\mu_{ii}(s) = -\sum_{j \neq i} \mu_{ij}(s)$$

Hence each row of the matrix $A(s)$ has zero sum.

5 Probability

Probability that the process goes into state j when it leaves state i

Given that the process is in state i at time s and it stays there until time $s + w$, the probability that it moves into state j when it leaves state i at time $s + w$ is:

$$\frac{\mu_{ij}(s + w)}{\lambda_i(s + w)} = \frac{\text{the force of transition from state } i \text{ to state } j \text{ at time } s + w}{\text{the total force out of state } i \text{ at time } s + w}$$

6.1 Integrated form of the Kolmogorov backward equations

Backward:
$$p_{ij}(s, t) = \sum_{l \neq i} \int_0^{t-s} p_{il}(s, s+w) \mu_{ik}(s+w) p_{kj}(s+w, t) dw \quad \text{when } i \neq j$$

The backward equation is obtained by considering the timing and nature of the first jump after time s . The duration spent in this initial state before jumping to another state (state k say) is denoted by w . The integral reflects the three stages involved:

1. remaining in state i from time s to time $s + w$
2. jumping from state i to state k at time $s + w$
3. moving from state k at time $s + w$ to state j at time t (possibly visiting other states along the way).

We then consider the possible values of w to obtain limits of 0 and $t - s$ for the integral, and we sum over all possible intermediate states k .

When $i = j$, the equation is:

$$p_{ii}(s, t) = \sum_{l \neq i} \int_0^{t-s} p_{il}(s, s+w) \mu_{ik}(s+w) p_{ki}(s+w, t) dw + p_{ii}(s, t)$$

The extra term here is to account for the possibility of staying in state i from time s to time t .

6.2 Integrated form of the Kolmogorov forward equations

Forward: $p_{ij}(s, t) = \sum_{k \neq j} \int_0^{t-s} p_{ik}(s, t-w) \mu_{kj}(t-w) p_{jj}(t-w, t) dw$

The forward equation is obtained by considering the timing and nature of the last jump before time t . The duration then spent in this final state (state j) before time t is denoted by w . The integral reflects the three stages involved:

- 1) moving from state i at time s to state k at time $t - w$ (possibly visiting other states along the way)
- 2) jumping from state k to state j at time $t - w$
- 3) remaining in state j from time $t - w$ to time t .

We then consider the possible values of w to obtain limits of 0 and $t - s$ for the integral and sum over all possible intermediate states k .

When $i = j$, the equation is:

$$p_{ii}(s, t) = \sum_{k \neq j} \int_0^{t-s} p_{ik}(s, t-w) \mu_{ki}(t-w) p_{ii}(t-w, t) dw + p_{ii}(s, t)$$

The extra term here is to account for the possibility of staying in state i from time s to time t .