INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M.Sc. SEM 3

Subject : Statistical and Risk Modelling - 3

Chapter: Unit 1 Chapter 1

Chapter Name: Introduction to stochastic processes

Today's Agenda

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    4. Continuous state space
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    1. General random walk

3. Poisson process

1.1 Introduction to Stochastic processes

Definition - A stochastic process is a model for a time-dependent random phenomenon. So, just as a single random variable describes a static random phenomenon, a stochastic process is a collection of random variables , X_t one for each time t in some set J.

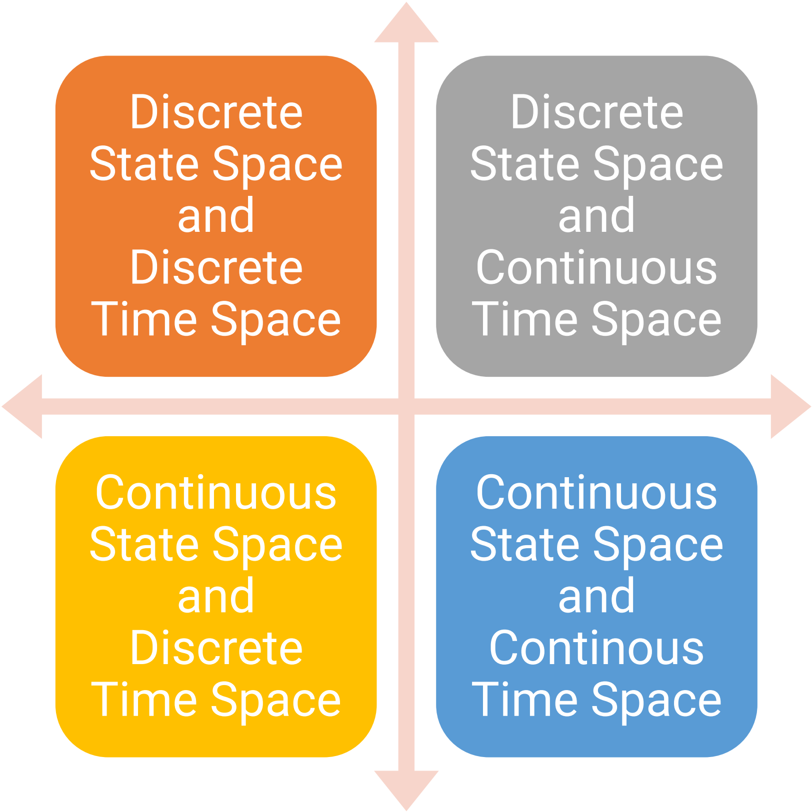
The process is denoted $\{ X_t : t \in J\}$.

The set of values that the random variables X_t are capable of taking is called the state space of the process, S .

For example, we might model the closing value of the NIFTY index by a stochastic process $\{X_t\}$. The random variable X_t models the value at the end of day t.

1.1 Introduction to Stochastic processes

There are 4 types of processes based on state space and time set :-



1.2 Discrete State Space and Discrete Time Space

Here is an example of a process with a discrete state space and discrete time changes.

A motor insurance company reviews the status of its customers yearly. Three levels of discount are possible (0, 25%, 40%) depending on the accident record of the driver.
In this case the appropriate state space is $S = \{0, 25, 40\}$ and the time set is $J = \{0, 1, 2,\ldots\}$ where each interval represents a year.

The time set often starts at 0 (whether continuous or discrete). Time 0 is taken to be the start, so that after one unit of time (day, minute etc) we have $t = 1$.

1.3 Discrete State Space with continuous time changes

A life insurance company classifies its policyholders as Healthy, Sick or Dead. Hence the state space $S = \{H, S, D\}$. As for the time set, it is natural to take $J = [0, \infty)$ as illness or death can occur at any time.
On the other hand, it may be sufficient to count time in units of days, thus using $J = \{0, 1, 2, \ldots\}$.

The given description indicates that the time set may either be continuous or discrete and may be recorded frequently. To address questions like "What is the probability that an individual, who is healthy at time s, becomes sick at time t?", it is essential to have a model capable of handling a continuous time domain.

1.4 Continuous state space

Claims of unpredictable amounts reach an insurance company at unpredictable times; the company needs to forecast the cumulative claims over [0, t] in order to assess the risk that it might not be able to meet its liabilities. It is standard practice to use $[0, \infty)$ both for S and J in this problem.

However, other choices are possible: claims come in units of a penny and do not really form a continuum.
Similarly, the intra-day arrival time of a claim is of little significance, so that {0, 1, 2,... } is a possible choice for J and/or S .

An important class of models having a continuous state space and a discrete time set is time series. Many economic and financial stochastic processes fall into this class, eg daily prices at the close of trading for a company's shares.

1.5 Mixed Process

Just because a stochastic process operates in continuous time does not mean that it cannot also change value at predetermined discrete instants; such processes are said to be of mixed type.

As an example, consider a pension scheme in which members have the option to retire on any birthday between ages 60 and 65. The number of people electing to take retirement at each year of age between 60 and 65 cannot be predicted exactly, nor can the time and number of deaths among active members. Hence the number of contributors to the pension scheme can be modelled as a stochastic process of mixed type with state space $S = \{1,2,3\}$ and time interval $J=[0, \infty)$. Decrements of random amounts will occur at fixed dates due to retirement as well as at random dates due to death.

So, this process is a combination (or mixture) of two processes:

- a stochastic process modelling the number of deaths, which has a discrete state space and a continuous time set
- a stochastic process modelling the number of retirements, which has a discrete state space and a discrete time set.

We model the total number of decrements and observe the initial number of members less the total number of decrements in $(0, t)$. This is a mixed process.

1.6 Counting Processes

Definition – A counting process is a stochastic process, X , in discrete or continuous time, whose state space S is the collection of natural numbers {0,1,2,...}, with the property that X (t) is a non-decreasing function of t .

White noise

White noise is a stochastic process that consists of a set of independent and identically distributed random variables. The random variables can be either discrete or continuous and the time set can be either discrete or continuous.

1.7 Sample Paths

Having selected a time set and a state space, it remains to define the process X_t : t ∈ J itself. This amounts to specifying the joint distribution of $X_{t1}, X_{t2}, \ldots, X_{tn}$ for all t1, t2, ..., tn in J and all integers n .

Sample paths

A joint realization of the random variables X_t for all t in J is called a sample path of the process; this is a function from J to S.

The properties of the sample paths of the process must match those observed in real life (at least in a statistical sense). If this is the case, the model is regarded as successful and can be used for prediction purposes. It is essential that at least the broad features of the real-life problem be reproduced by the model;

1.8 Stationarity

Strict stationarity
A stochastic process is said to be stationary, or strictly stationary, if the joint distributions of X_{t1}, X_{t2}, …. , X_{tn} and X_{k+t1}, X_{k+t2}, …. , X_{k+tn} are identical for all t1, t2, …., tn and k+t1, k+t2, …., k+tn in J and all integers n.
This means that the statistical properties of the process remain unchanged as time elapses.

Strict stationarity is a stringent requirement which may be difficult to test fully in real life.

Let's take the example ahead, with the three states Healthy, Sick and Dead. One would certainly not use a strictly stationary process in this situation, as the probability of being alive in 10 years' time should depend on the age of the individual.
The probability of a life aged 50 surviving to age 60 is not likely to be the same as the probability of a life aged 75 surviving to age 85.

1.8 Stationarity

Strict stationarity is a stringent requirement which may be difficult to test fully in real life. For this reason, another condition, known as weak stationarity is also in use.

Weak stationarity

A process is weakly stationary if:

E(X_t) is constant for all t , and

cov(X_t, X_{t+k}) depends only on the lag, k .

To be weakly stationary a process must pass both these tests. If it fails either of the tests, then it is not weakly stationary. So, when checking for stationarity, start with the easier condition (the mean), then check the covariances.

If a process is strictly stationary, then it is also weakly stationary. However, a weakly stationary process is not necessarily strictly stationary.

1.9 Increments

Increments

An increment of a process is the amount by which its value changes over a period of time, eg. X_{t+u} - X_t (where u > 0).

Independent increments

A process X_t is said to have independent increments if for all t and every u > 0 the increment X_{t+u} - X_t is independent of all the past of the process $\{X_s: 0 \leq s \leq t\}$.

Question

(i) Define an increment of a process. [1]

The rate of mortality in a certain population at ages over exact age 30 years, $h(30 + u)$, is described by the process:

$$h(30 + u) = B(1 + \gamma)^{u} \quad , u \geq 0$$

where B and γ are constants.

(ii) Show that the increments of the process $\log[h(30 + u)]$ are stationary. [3]
[Total 4]

Solution

(i) An increment of a process is the amount by which its value increases over a period of time, for example $X(u + t) - X(u)$ where $t > 0$. +1

[1]

(ii) EITHER

$$\log[h(30+u+t)] - \log[h(30+u)] = \log[B(1+\gamma)^{u+t}] - \log[B(1+\gamma)^{u}] \quad +1$$

$$= \log B + (u+t)\log(1+\gamma) - (\log B + u\log(1+\gamma)). \quad +½$$

$$= t\log(1+\gamma) \quad +½$$

The increment thus depends on t, the duration of the process but not on u, hence the process is stationary. +1

OR

$$\log h(30+u) = \log B + u\log(1+\gamma) \quad +1$$

$$\frac{d}{du}\log h(30+u) = \log(1+\gamma) \quad +1$$

which is constant and does not depend on u, so the process is stationary. +1

[3]

[Total 4]

1.2 Markov Process

A stochastic process is called a Markov process if it exhibits the Markov property.

Stated precisely the Markov property reads:
P$[X_t \epsilon A \big| X_{s_1} = x_1, X_{s_2} = x_2, X_{s_3} = x_3 \ldots X_{s_n} = x_n, X_s = x \big] = P[X_t \epsilon A | X_s = x]$ for all times $s_1 < s_2 < s_3 \ldots s_n < s < t, all\ states\ x_1, x_2 \ldots x_n\ and\ x\ is\ S\ and\ all\ subsets\ A\ of\ S.$
This is called this Markov Property

A process with independent increments has the Markov property, but a Markov process does not necessarily have independent increments.

Proof:
P$[X_t \epsilon A | X_{S_!} = x_!, X_{s_2} = x_2, \ldots X_{S_n} = x_n, X_s = X]$
= P$[X_t - X_S + x \epsilon A \big| X_{S_!} = x_!, X_{s_2} = x_2, \ldots X_{S_n} = x_n, X_s = X \big]$
= P$[X_t - X_S + x \epsilon A | X_s = X]$
= P$[X_t \epsilon A | X_s = X]$

1.10 Filtrations

The following structures underlie any stochastic process X_t:

- a sample space Ω : each outcome “w” in Ω determines a sample path X_t (w).
- a set of events F : this is a collection of events, by which is meant subsets of Ω , to which a probability can be attached
- for each time t , a smaller collection of events F_t c F : this is the set of those events whose truth or otherwise are known at time t . In other words, an event A is in F_t if it depends only on X_s , 0 ≤ s ≤ t.

As t increases, so does F_t: F_t c F_u, t ≤ u. Taken collectively, the family $(F_t)_{t\geq 0}$ is known as the (natural) filtration associated with the stochastic process X_t, t ≥ 0 ; it describes the information gained by observing the process or the internal history of X_t up to time t .

This is the key thing to know about the (natural) filtration – that F_t gives the history of the process up to time t.

The process X_t can be said to have the Markov property if: P[X_t ≤ x | F_s] = P[X_t ≤ x | X_s] for all 0 ≤ s ≤ t.

2 Random Walk

Simple Random Walk
This is defined as $X_n = Y_1 + Y_2 + \cdots Y_n$ where the random variables Yj (the steps of the walk) are mutually independent with the common probability distribution:
P$[Y_j = 1] = p$, $P[Y_j = -1] = 1 - p$

The Markov property holds because the process has independent increments. The transition graph and the transition matrix are infinite.

A random walk has independent increments, so it has the Markov property.

2.1 General Random Walk

Start with a sequence of IID random variables $Y_!$ ..., Y_j .... and define the process:

$$X_n = \sum_{j=1} Y_j$$

with initial condition $X_o = 0$.
The formula for the random walk can also be written as:

$$X_n = X_{n-1} + Y_n \, , n = 1,2,3,4$$

This is a process with stationary independent increments, and thus a discrete-time Markov process. It is known as a general random walk. The process is not even weakly stationary, as its mean and variance are both proportional to n .

The log of the closing value of the FTSE100 index could be modelled by a general random walk, the value one day being the value on the previous day plus some random adjustment.

3 Poisson Process

A Poisson process with rate λ is a continuous-time integer-valued process N_t , t > 0 , with the following properties:

(i) $N_o = 0$

(ii) N_t has independent increments

(iii) N_t has Poisson distributed stationary increments:

$$P[N_t - N_s = n] = \frac{[\lambda(t-s)^n]e^{-\lambda(t-s)}}{n!}, s < t, n = 0, 1 \ldots$$

i.e. $N_t - N_s \sim Poisson\ (\lambda(t-s)) for\ 0 \leq s < t$

This is a Markov jump process with state space S = {0,1,2..} . It is not stationary: as is the case for the random walk, both the mean and variance increase linearly with time.

The process counts the number of events (that are occurring at a rate λ per unit time) that occur between time s and time t . It is an example of a counting process.

Question

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For each of the following processes:

- simple random walk
- Markov jump process
- compound Poisson process
- Markov chain
- counting process

(a) State whether the state space is discrete, continuous, or can be either.

(b) State whether the time set is discrete, continuous, or can be either.
[5]

Solution

Process	*State space*	*Time set*	
Simple random walk	Discrete	Discrete	[+1]
Markov jump process	Discrete	Continuous	[+1]
Compound Poisson process	Either	Continuous	[+1]
Markov chain	Discrete	Discrete	[+1]
Counting process	Discrete	Either	[+1]