60mks

~~CZZ~~

CS 2

SRM - 3

Math heavy (integral)

Unit 1
Intro. to Stochastic process
5-10m
Theory

(25% each)

2
Markov chains
15-20M

Practice all Qs

3
Markov jump process *
10-15M
time independent & Poisson model

4
Markov Process
10-15m
time dependent

Ch. - 1 - Intro. to Stochastic Process

- Intro
- Types of Stochastic process
- Mixed Stochastic process
- Sample path
- Stationarity
- Increments
- Filtration
- Markov property
- Ex. of Stochastic processes

trivia

Stochastic → derived from Greek word Stokhos → guess / try to aim

Deterministic → Bond 100 → Exp. value 1 yr 105

Expected return = 5%

or

$\boxed{\text{Stochastic}}$

↓

Ex. - Stock returns, inflation rates, mortality rates.

Share A 100 → 1 yr

Exp. stock mkt. return = 8%

→ randomness in atleast 1 input → Output will be random / stochastic

Stochastic Process

ex. height, gender, etc

→ It is a collect^n of random variables (r.v.s) 'X_t' for each time 't' in a time set 'J' $[t \in J]$

→ It is a time dependent phenomenon

Ex - Height of all students in SRM 3 class

↓ Not a stochastic process

→ collect^n of r.v.s ✓

× time dependent

→ Height of a person every month from age 10 to 18.

→ Stochastic process

Ex - → Stock price movements

→ Weather forecast / recordings

Types of Stochastic processes

↓

① State space

→ Set of values that r.v. can take

Ex. Stock price → {99, 100, 101, 102 ....}

② Time set / Time Domain

→ Collection of times at which process values are recorded.

Discrete → particular value → finite set of values

↓

numeric or classification

Time Domain

State Space	Discrete	Continuous
Discrete	1 No Claims Discount model (NCD)	2 HSD → Health - Sickness - Death model (Life Ins. Co.)
Continuous	3 Share price at day end	4 Share price tracked continuously

Continuous

→ infinite, ex → rational nos. → always be numeric

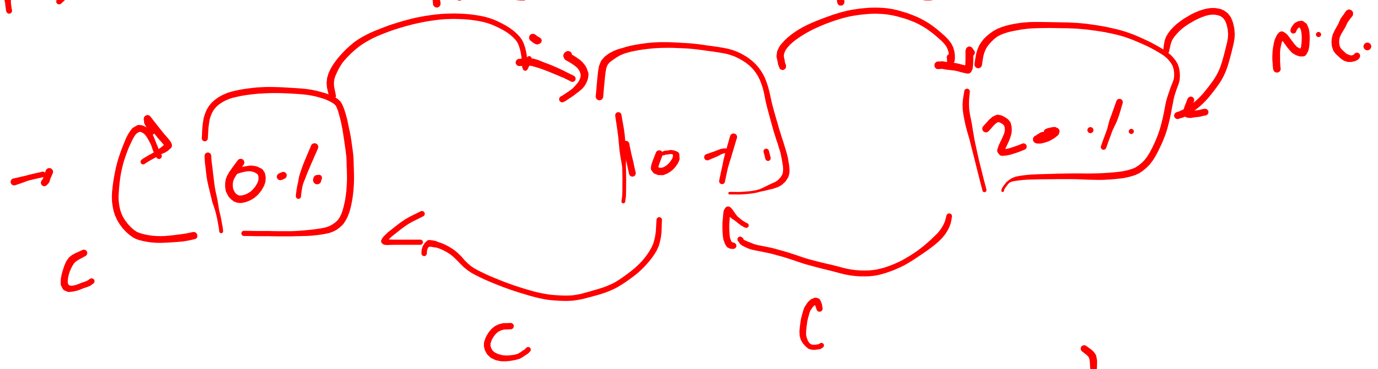
NCD

↓

Motor Ins.

↓

Prem. ₹ 5000

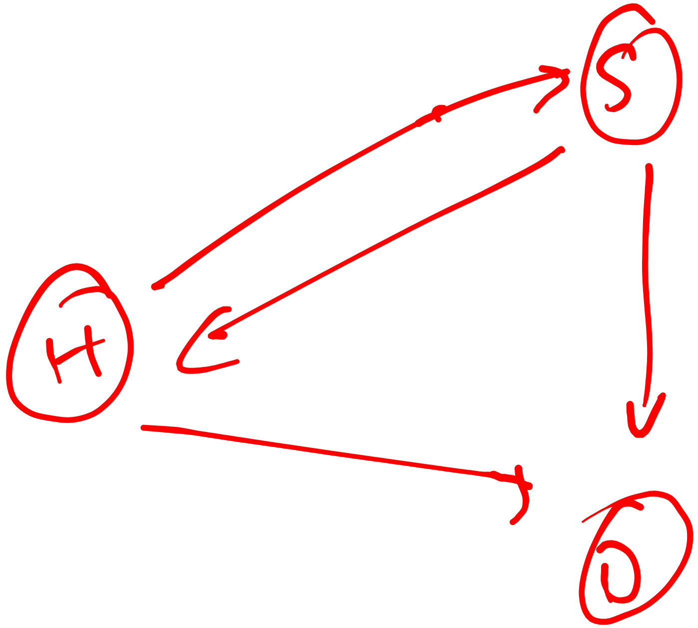


→ S.S. could be numeric or classifies.

Discrete S.S. → {0%, 10%, 20%}

Discrete T.D. → year end.

HSD



S.S. = {H, S, D}

T.D. → continuous

Discrete S.S.
Continuous T.D.
↓
because transitions can happen at any time.

Sample path

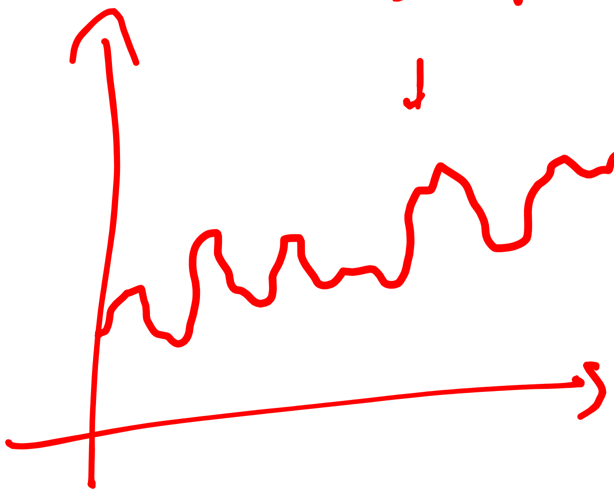
→ Joint realizatⁿ of r.v. for all 't' in the time set 'T'.

Ex → Closing stock price → {2400, 2405, 2378 . . . }

→ NCO → {0%, 10%, 20%, 20%, 10% . . . }

→ Simply the outcome sequence of particular set of experiments.

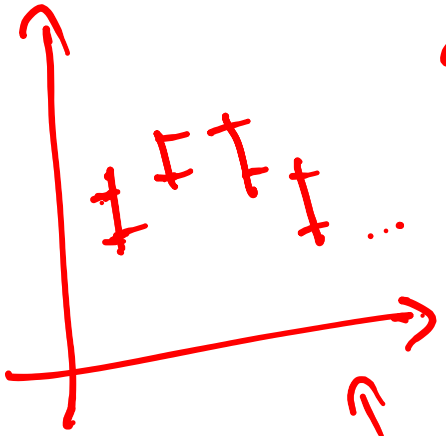
Sample path



Stock price tracked continuously

Sample path? → censoring info. → Not a sample path.

30min →



OHLC → Open High Low Close

Mixed Stochastic Process

S.S.	T.D.
D/C	D
D/C	C

S.S.	T.D.
D	D/C
C	D/C

Ex. → Income of a car garage. → 31/3 → T.D. + Discrete

Continuous ← Repairs

Selling old cars [₹ 5 lacs] → Discrete

Not graph
A stochastic process

Ins. Co. → Mediclaim → Ex. → 5 lacs.

Ins. Co. → Motor Ins. → Ex. → Reimburse 100%

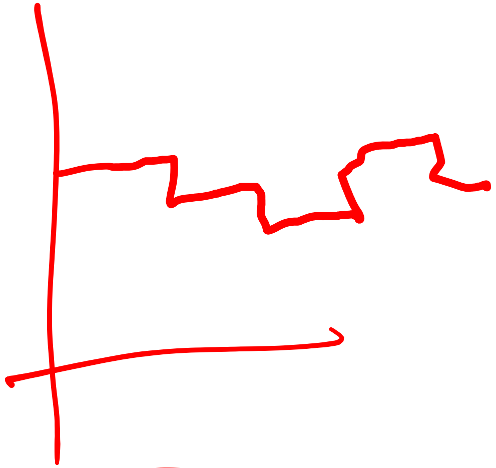
Pension liab. → Cont. SS.

Ex. 2 → No. of members in an E'er retirement scheme.

Discrete S.S. = {0, 1, 2 .....}

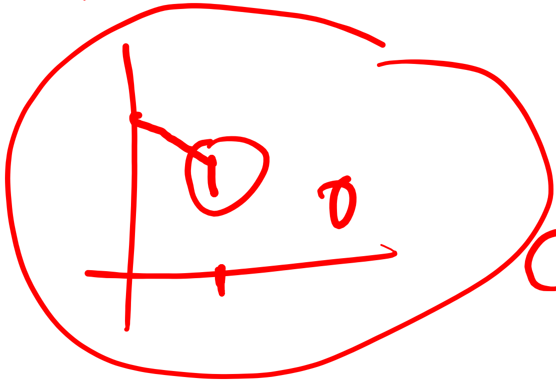
$\boxed{60}$ → a s.l.

→ 5%.



Retirement age → 60 → Discrete T.D.

Death, retrenchment, superannuat? (retirement before retirement age)



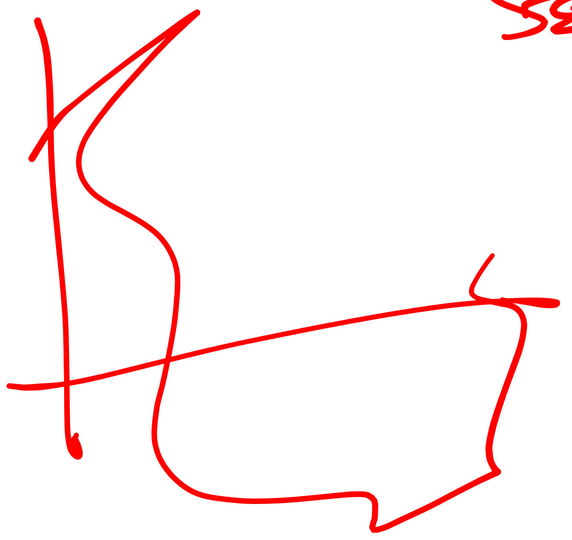
Continuous T.D.

Increment → Amt. by which value of process changes over a period of time.

ie. $X_{t+u} - X_t$ is an increment over time, provided $u > 0$

Independent increments → If, for all 't', & all $u > 0$, the increment $X_{t+u} - X_t$ is independent of all past process, ie. independent of 'X_s' where $0 \le s \le t$.

Ex.



~~Set~~ Stock prices

↓

t t t < Random

Independent of past.

F: ~~ltred~~

Filtration

Conditional prob.

$P[A/B]$

$F_t \to P[A_R \geq 100] \to P[A_R \geq 100 / 70, 22, 103, \ldots]$

past history

$\downarrow$

$P[m_S \geq 100]$

$P[X_k \in A / X_1, X_2, X_3 \ldots X_{k-1}]$

$= P[X_k \in A / F_k]$

term to represent past history.

$X_t \in A$

- Every stochastic process has a collectⁿ of events known at time 't' i.e. past history
- It is a theoretical difficulty to list all these values for a continuous time process
- The filtratⁿ notation 'F_t' is used to describe all info. gained upto time 't'.

Stationarity → very desirable property.

→ If statistical properties of a process do not vary over time, the process is stationary.

$(X_1, X_2, X_3, \ldots)$

Strict stationarity → Impractical

→ Joint distribution of X_t, X_{t+1}, X_{t+2} is the same as $X_{t+k}, X_{t+k+1}, X_{t+k+2}$ -

Weak stationarity

→ $E(X_t)$ → constant over time

→ ↓

Ex → $P[\underline{X_1 > 100} \,\&\, \underline{X_2 < 150} \,\&\, \underline{X_3 < 200}]$

$= P[\underline{X_7 > 100} \,\&\, \underline{X_8 < 150} \,\&\, \underline{X_9 < 200}]$

All permutat's & combinat's of this should be ~~true~~ true.

Ex → No. of heads in a coin toss →
or
No. on top of a dice after a roll

} Universal factual events

Ex - White Noise Process

$cov(X_t, X_t) = cov(X_{t+k}, X_{t+k})$

$\rightarrow$ Covariance b/w 2 variables only depends on the lag

i.e. $cov(X_t, X_{t+k}) \rightarrow$ only depend upon 'k'

$\rightarrow$ Not depend upon 't'.

$V(X_t) = V(X_{t+k})$

Variance is constant over time.

~~Heteros~~ Heteroscedastic $\times$

ex. lag 3

$Cov(X_1, X_4) = Cov(X_8, X_{11}) = \ldots$

~~k = 0~~, lag 0

$cov(X_t, X_t) = cov(X_{t+k}, X_{t+k})$

$V(X_t) = V(X_{t+k}) \rightarrow$ constant

Q:

i) Define an increment of process.

$31 \to h(31) = \ldots$

$35 \to h(35)$

Rate of mortality for people over age of 30.

i.e. $h(30+u) = B(1+r)^u , \quad u \geq 0$

B and r are constants.

ii) Show that increments of the process $\log[h(30+u)]$ are stationary.

$f_x(1 \ldots f(u))$

$$\log A \cdot B = \log A + \log B$$

$$\log A^B = B \cdot \log A$$

Sol^n →

$$X_t = \log[n(30+t)] = \log[B(1+r)^t]$$

Increment

$$X_{t+1} - X_t = \log[B(1+r)^{t+1}] - \log[B(1+r)^t]$$

$$= \log B + \log(1+r)^{t+1} - \log B - \log(1+r)^t$$

$$= (t+1) \cdot \log(1+r) - t \cdot \log(1+r)$$

$$= \underline{\log(1+r)} \rightarrow \underline{\text{constant}}$$

→ Mean → constant

Var → 0

∴ Process is stationary.

V(constant) = 0

↓

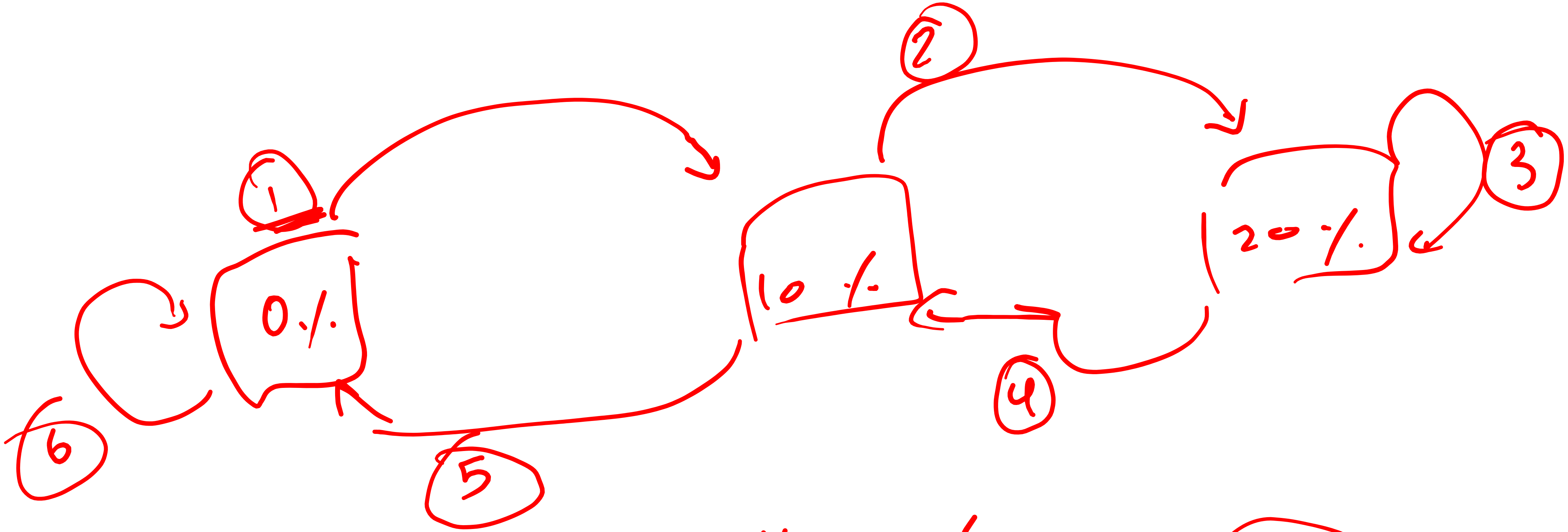
fluctuation

Markov property:

$$P(X_{t+1} \in A / F_t) = P(X_{t+1} \in A / X_t)$$

→ If future development of a process can be predicted from its current state only, i.e. from latest available value w/o any reference to its past history, the process is said to follow the Markov property.

Ex. → NCD → prob of claim = p



$X_1 = 0\%$

$P(X_2 = 10\%) = 1-p$

$X_6 = 0\%$

$P(X_7 = 10\%) = 1-p$

Ex. → If it rained yesterday, prob. of rain today = 40%. → prob. of no rain = 60%.

R or NR

If it did not rain yesterday, prob. of rain today = 30%. → follows the Markov Property

Past is irrelevant

M	T	W	T	F
R	R	R	NR	
NR	NR	NR	NR	

← P(R) = 0.3

← P(R) = 0.3

only latest available info / current state influences future prob.

Prob. of ~~switching~~ win after ~~(switching)~~

Stochastic experiment | Select? → Stay or switch

→ Is this a Markov process?

Step 1 → Select

Step 2 → Host eliminates a door

Step 3 → $\boxed{\text{Door X or Door Y}}$ → Markov prop.?

→ dependent upon past → Not a Markov process

Application of Markov:

1 2 3 ...

- → Finance → Credit rating agency
- → Chemistry → Bonds
- → Physics | Mechanical Engineering
- → Politics | Social Science
- → Games → Snakes & ladder
- → AI | ML
  - → Google Pagerank

Nodes

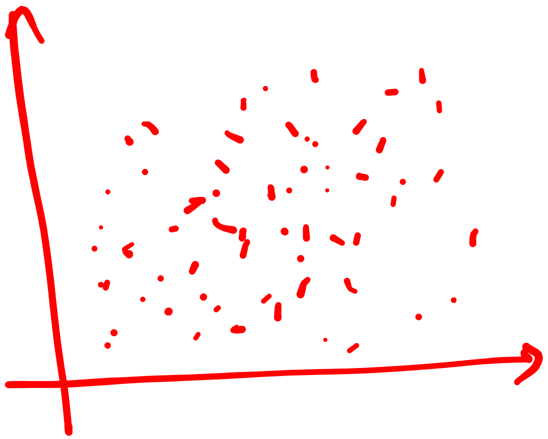
Links → Structure

Ex. of Stochastic process

① White Noise Process

→ A stochastic process that consists of independent & identically distributed (i.i.d) r.v.s.

Ex. R.V. of top no. on roll of a dice



- Strictly stationary process.
- Has the Markov property.

→ To model errors in your Process.

$$X_t = \underline{1.02}\, X_{t-1} + \boxed{e_t}$$

$\boxed{e_t}$ → follow WN?

→ random fluctuat²

error term

$E(e_t) = 0$

$V(e_t)$ → concerned

→ It could be $N(0, \sigma^2)$

-36 36

99.73% the time

Q.

1) Define increment of process

$h(30) = B(1+r)^0$

$= B$

Rate of mortality for people over 30

$\rightarrow h(30+u)$

$h(30+u) = B(1+r)^u \quad , \quad u \geq 0$

B & r are constants

2) Show increments of process $\log[h(30+u)]$ are stationary.

$X_{t+1} - X_t$

$$X_u = \log[h(30+u)] = \log[B(1+r)^u]$$

Increment

$$X_{t+1} - X_t = \log[B(1+r)^{t+1}] - \log[B(1+r)^t]$$

Basics

$$\log A.B = \log A + \log B$$

$$\log A^n = n.\log A$$

$$X_{t+1} - X_t = \log B + \log(1+r)^{t+1} - \log B - \log(1+r)^t$$

$$= (t+1).\log(1+r) - t.\log(1+r)$$

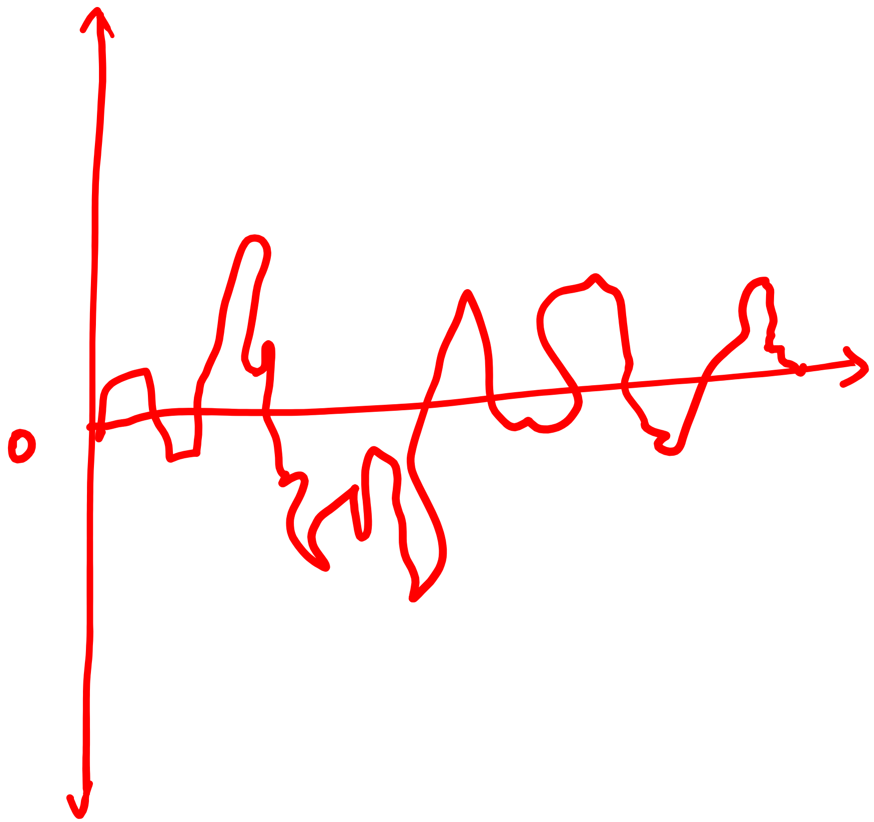
$I : X_{t+1} - X_t = \log(1+r) \rightarrow$ constant

$E(I) \rightarrow$ constant

$V(I) = 0$

$\therefore$ Process is stationary.

② General Random Walk



$$\boxed{X_n = \sum_{i=1}^{n} Y_i} \quad \& \quad X_0 = 0.$$

$Y_i \to$ WNP

Ex. $\to$ futures position in stock mkt.

Daily chgs. $\to$ WNP

Final P/L posit^n $= \sum$ WNP

$=$ Random Walk.

$X_0 = 0$

$X_1 = Y_1$

$X_2 = Y_1 + Y_2 = X_1 + Y_2$

$X_3 = Y_1 + Y_2 + Y_3 = X_2 + Y_3$

....

Increments → Independent increments

↓

follows the Markov property

Any process with Independent Increments follow the Markov property

Simple Random Walk

→ Y_i can only take 2 values i.e. $\boxed{-1 \ \& \ +1}$

→ Ex. Coin toss / $\boxed{\text{win/loss}}$

Simple Symmetric R.W.

→ Simple R.W. (+) $P(Y_i = 1) = P(Y_i = -1)$

$= 0.5$

Ex. Coin toss : H → +1 Rs.

T → −1 Rs.

③ Poisson Process

N_t Poi → $\frac{e^{-\lambda} \cdot \lambda^n}{n!}$

→ A Poisson process with rate λ is a continuous time integer valued process 'N_t', $t \geq 0$ with foll. properties –

① → $N_0 = 0$

② N_t has independent increments

③ $N_t - N_s \sim Poi[(t-s)\lambda]$

i.e. $P[N_t - N_s = n] = \frac{e^{-(t-s)\lambda} [(t-s)\lambda]^n}{n!}$

Ex → No. of claims in a health insurance office

$N_0 = 0$, no. of claims everyday ~ Poi(4)

Ex.

↓

Poisson process

↓

used in modelling

no. of claims / frequency

$$E(X) = \lambda$$
$$V(X) = \lambda$$

→ Has the Markov property

→ Stationary? → Not a stationary process

↓

aggregation → E( ) & V( ) ↑

→ Aggregation process → $E(N_t)$ & $V(N_t)$ both increase with time

④ Compound Poisson process

$$X_t = \sum_{i=1}^{N_t} Y_i \quad , \quad t \geq 0$$

Y_i → represents severity

& Y_i is an iid rv

$$N_t - N_s \sim Poi[\lambda(t-s)]$$

Ex → Total claims in a health ins. office

freq ~ Poi process

Severity ~ iid r.v.

Is has the Markov property?

Yes

Freq ~ Poi process → independent increments
(t)

Severity (Y_i) → iid r.v. → independent increments
↓
It follows Markov property

→ Independent increment

$X_t = Y_1 + Y_2 + Y_3 + \dots Y_{N_t}$

Stationary?

No $\rightarrow$ same reasons as Poisson process.

If $Y_i = 1$, then Compound Poi process is same as Poisson process.

Exam Style

Poi → $\frac{e^{-\lambda} \cdot \lambda^{n}}{n!}$

i) Define Poisson process.

ii) Town → Bus stop → every 15 mins. 1 bus arrives

Assume that no. of buses ~ Poisson process.

Mr. Bean → Bus stop at 12 midday.

a) Prob. that first bus will NOT arrive until 1 p.m. → $\boxed{0.0183}$

b) First bus arrives at 1.10pm, but it was full.
How much longer is Mr. Bean expected to wait for the next bus? → $\boxed{\text{15 mins.}}$ → because of independent increments

c) Prob. that at least 2 buses arrive b/w 1.10 & 1.20 pm.

ii) a) Prob. of no bus in 1 hr. :

No. of buses in 15 mins. $\sim Poi(1)$

No. of buses per min $\sim Poi(1/15)$

" " " per hour $\sim Poi(4)$

$$P(X=0) = \frac{e^{-4} \cdot 4^0}{0!} = e^{-4} = \underline{\underline{0.0183}}$$

$\underline{1.83\%}$

c) ~~No. of buses that~~ Prob. of at least 2 buses b/w 1.10 & 1.20 pm

No. of buses in 10 mins. $\sim$ Poi$\left(\frac{1}{15} \times 10 = \frac{2}{3}\right)$

$$P(X \geq 2) = 1 - P(X < 2)$$

$$= 1 - \left[P(X=0) + P(X=1)\right]$$

$$= 1 - \frac{e^{-2/3} \cdot (2/3)^0}{0!} + e^{-2/3} \cdot 2/3$$

$$= 0.1443$$

1) $X, Y, Z \sim \text{Exp}(x, y, z)$

$\min(X, Y, Z) \sim \text{Exp.}$ & $\text{par} = x + y + z$.

---

2)

Type of vehicle	Rate	Toll (₹)
Motorbike	2 per min.	1
Car	5 per min.	2
Goods vehicle	1.5 per min.	5

Toll → motorbike → ₹ 1, car → ₹ 2

goods vehicle → ₹ 5

~~Assumes vehicles~~ no. of

Assume : no. of vehicles ~ Poisson process.

① State name of stochastic process that describes total value of tolls collected. → Compound Poisson process.

② Calculate expected value of tolls collected per hour. → 1,170

③ Engineer : no more than 2 goods vehicles can come on the bridge in a minute.

Engineer → no more than 2 goods vehicles in a min.

a) Calculate prob. of more than 2 goods vehicles in a min.

b) Calculate expected loss due to this. per minute.

$\boxed{6 \text{ mins}}$

4 Calc. prob. that exactly £4 is earned in a minute. → $\boxed{3 \text{ mins}}$

2

	M	C	GV	
per min.	2	5	1.5	
per hr.	120	300	90	
Rate	1	2	5	
Toll	120	600	450	1170

3

a) $P(X>2) = 1 - P[X \leq 2]$

$= 1 - [P(X=0) + P(X=1) + P(X=2)]$

$= 1 - \left(e^{-1.5} + e^{-1.5} \cdot 1.5 + \frac{e^{-1.5} \cdot 1.5^2}{2}\right)$

$= 0.1912$

b)

Exp. loss (purely from goods vehicles)

$= \text{Exp. toll currently} - \text{Exp. toll after the change}$ ☆

$= 1.5 \times 5 - \left[\sum_{x=0}^{\infty} x \cdot P(X=x) \right]$

$= 1.5 \times 5 - \Big[P(X=0) \times 0$

$+ P(X=1) \times \cancel{10}\ 5$

$+ P(X=2) \times 10$

$+ P(X>2) \times 10 \Big]$ → £10 will be collected even when more than 2 goods vehicles pass.

$= \underline{1.4043}$

Q. 2

i) Prove memoryless property of Exp. dist^n → Unit 3

3 independent exp. dist^{ns}: X, Y, Z

$X \to x$, $Y \to y$, $Z \to z$

ii) Show that $\min(X, Y, Z)$ is also an exponential dist^n & give its parameter.

iii) Toll bridge → no. of vehicles ~ Poi. process

Motorcycle	~	2 per min.	Motorcycle	→ 1 ₹
Car	~	5 per min	Car	→ 2 ₹
Goods vehicle	~	1.5 per min.	Goods veh.	→ 5 ₹

Name of process that can be used to determine total value of tolls collected. → Compound Poisson process.

v) Calc. exp. value of tolls per hour.

$= 2 \times 60 \times 1 + 5 \times 60 \times 2 + 1.5 \times 60 \times 5$

$= £1,170$

vi) → Engineer → bridge can't take more than 2 goods vehicles at a time (in a given minute)

→ Calc. prob. that more than 2 goods vehicles arrive in a given minute, and calc. exp. loss due to the change.

$X \sim Exp(m)$, $Y \sim Exp(y)$, $Z \sim Exp(z)$

$\min(X, Y, Z) \sim Exp[\,?\,]$

$\boxed{P[X \le m]}$

Hint → Use cdf of Exp. distⁿ

$P[\min(X, Y, Z) \le t] = 1 - P[\min(X, Y, Z) > t]$

$= 1 - P[X, Y, Z > t]$

$= 1 - P(X > t, Y > t, Z > t)$

$= 1 - P(X > t) \cdot P(Y > t) \, P(Z > t)$

$$P(X \le t) = 1 - e^{-xt}$$

$$P(X > t) = 1 - \left[1 - e^{-xt}\right] = \underline{e^{-xt}}$$

$$P[\min(X, Y, Z) \le t] = 1 - e^{-xt} \cdot e^{-yt} \cdot e^{-zt}$$

$$= \underline{1 - e^{-(x+y+z)t}}$$

$\downarrow$

cdf of exp. dist^n

$$\underline{\text{Parameter} = x + y + z}$$

(4) → P(toll p.m = 4) →

m →	1	2
C →	2	5
G →	5	1.5

4 m	2 m	0 m
0 C	1 C	2 C
0 G	0 G	0 G

$$P(m=4, C=0, G=0) + P(m=2, C=1, G=0) + P[m=0, C=2, G=0]$$

$$\frac{e^{-2}\cdot 2^4}{4!} \times e^{-5}\cdot e^{-1.5} + \frac{e^{-2}\cdot 2^2}{2!}\cdot e^{-5}\cdot 5 \cdot e^{-1.5} + e^{-2}\cdot \frac{e^{-5}\cdot 5^2}{2}\cdot e^{-1.5}$$

$$= 0.0047 = 0.47\%$$