INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: M.Sc. SEM 3

Subject : Statistical Techniques and Risk Management 3

Chapter: Unit 3 Chapter 1

Chapter Name: Markov Jump Process & Two State Model

Today’s Agenda

1. Markov Jump Process
2. The two-state Markov model
3. Assumptions underlying the model
4. Survival Probabilities
5. Statistics and Real-life Application

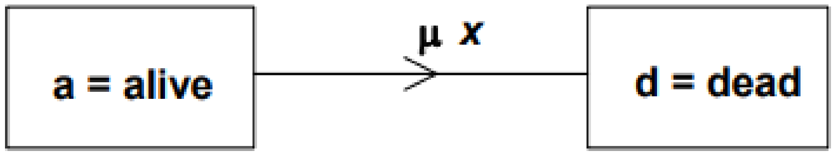
1 Markov jump process

Markov jump process
A continuous-time Markov process X_t , t ≥ 0 with a discrete (i.e. finite or countable) state space S is called a Markov jump process.

Example
- Health-sickness-death model

2 The two-state Markov model

The two-state model is illustrated in the figure below. There is an alive state and a dead state, with transitions in one direction only. This is an example of a Markov jump process.



We define a transition probability ${}_tq_x$ where:
$\boldsymbol{{}_tq_x}$ **= P[person in the dead state at age x+t | in the alive state at age x]**
And an occupancy or survival probability ${}_tp_x$ where:
$\boldsymbol{{}_tp_x}$ **= P[person in the alive state at age x+t | in the alive state at age x]**

The probability that a life alive at a given age will be dead at any subsequent age is governed by the **age-dependent transition intensity** $\boldsymbol{\mu_{x+t}}$ (t ≥ 0).

3 Assumptions underlying the model

There are three assumptions underlying the simple two-state model.

- Assumption 1

The probabilities that a life at any given age will be found in either state at any subsequent age depend only on the ages involved and, on the state currently occupied. This is the Markov assumption.

- Assumption 2

For a short-time interval of length dt: $\boldsymbol{{}_{dt}q_{x+t} = \mu_{x+t}\,\mathrm{dt} + o(\mathrm{dt})}$ (t ≥ 0).
In other words, the probability of dying in a very short time interval of length dt is equal to the transition intensity multiplied by the time interval, plus a small correction term. This is equivalent to assuming that $\boldsymbol{{}_{dt}q_{x+t} \approx \mu_{x+t}\,\mathrm{dt}}$

- Assumption 3

μ_{x+t} is a constant μ for 0 ≤ t < 1.

4 Survival Probabilities

Since we have specified the model in terms of a transition intensity, we must see how to compute transition probabilities.

Consider the survival probability ${}_{t+dt}p_x$, and condition on the state occupied at age $x+t$.
Here we are thinking about the probability of surviving from age x to $x+t$ and onwards from age $x+t$ to $x+t+dt$.

By the Markov assumption (Assumption 1), nothing else affects the probabilities of death or survival after age $x+\boldsymbol{t}$

$$\begin{aligned}
{}_{t+dt}p_x = {}&{}_tp_x \times P\,[\text{Alive at } x+t+dt \mid \text{ Alive at } x+t] \\
&+ {}_tq_x \times P\,[\text{Alive at } x+t+dt \mid \text{ Dead at } x+t] \\
&= ({}_tp_x \times {}_{dt}p_{x+t}) + ({}_tq_x \times 0) \\
&= {}_tp_x \times (1-\mu_{x+t}dt + o(dt))
\end{aligned}$$

The last equality follows from Assumption 2.

4 Survival Probabilities

Therefore:

$$\frac{\partial}{\partial t}\,{}_t p_x = \lim_{dt \to 0^+} \frac{{}_{t+dt}p_x - {}_t p_x}{dt}$$

$$= -\,{}_t p_x\ \mu_{x+t} + \lim_{dt \to 0+} \frac{o(dt)}{dt}$$

$$= -\,{}_t p_x \mu_{x+t}$$

So:

$${}_t p_x = \exp\left(-\int_0^t \mu_{x+s} ds\right)$$

This is an important result which gives the relationship between survival probabilities and force of mortality.

5 Statistics and Real-life application

We suppose that we observe a total of N lives during some finite period of observation, between the ages of x and x + 1. We do assume that all N lives are identical and statistically independent.

For i = 1, … , N define:

- x + a_i to be the age at which observation of the i th life starts
- x + b_i to be the age at which observation of the i th life must cease if the life survives to that age.
- x + b_i will be either x + 1, or the age of the i th life when the investigation ends, whichever is smaller.

For simplicity we consider Type I censoring. The approach can be extended to more realistic forms of censoring.

5 Statistics and Real-life application

Define a random variable D_i as follows:

$$D_i = \begin{cases} 1 & \text{if the } i \text{ th life is observed to die} \\ 0 & \text{if the } i \text{ th life is not observed to die} \end{cases}$$

D_i is an example of an indicator random variable; it indicates the occurrence of death.

Define a random variable T_i as follows:
$x + T_i =$ the age at which observation of the i th life ends

Notice that D_i and T_i are not independent, since:

$$D_i = 0 \Leftrightarrow T_i = b_i$$

ie if no death has been observed, the life must have survived to $x + b_i$.

$$D_i = 1 \Leftrightarrow a_i < T_i < b_i$$

ie an observed death must have occurred between $x + a_i$ and $x + b_i$.

5 Statistics and Real-life application

It will often be useful to work with the time spent under observation, so define:

$$V_i = T_i - a_i$$

V_i is called the waiting time. It has a mixed distribution, with a probability mass at the point $b_i - a_i$

A mixed distribution has a discrete and a continuous part.

5 Statistics and Real-life application

Joint density function

The pair (D_i, V_i) comprise a statistic, meaning that the outcome of our observation is a sample (d_i, v_i) drawn from the distribution of (D_i, V_i).
Let $f_i(d_i, v_i)$ be the joint distribution of (D_i, V_i).
It is easily written down by considering the two cases $D_i = 0$ and $D_i = 1$.

$$f_i(d_i, v_i) = \begin{cases} {}_{b_i - a_i}p_{x+a_i} & (d_i = 0) \\ {}_{v_i}p_{x+a_i} \cdot \mu_{x+a_i+v_i} & (d_i = 1) \end{cases}$$

$$= \exp\left(-\int_0^{v_i} \mu_{x+a_i+t} dt\right) \mu_{x+a_i+v_i}^{d_i}$$

Now assume that $\mu_{\mathrm{x}+t}$ is a constant μ for $0 \leq t < 1$ (this is the first time we have needed Assumption 3) and so $f_i(d_i, v_i)$ takes on the simple form:

$$f_i(d_i, v_i) = e^{-\mu v_i} \mu^{d_i}$$

5 Statistics and Real-life application

The maximum likelihood estimator

We have already seen that the joint probability function of all the (D_i, V_i) is:

$$\prod_{i=1}^{N} e^{-\mu v_i} \mu^{d_i} = e^{-\mu(v_1+\cdots+v_N)} \mu^{d_1+\cdots+d_N} = e^{-\mu v} \mu^d$$

where $d = \sum_{i=1}^{N} d_i$ and $v = \sum_{i=1}^{N} v_i$

This probability function immediately furnishes the likelihood for μ :

$$L(\mu; d, v) = e^{-\mu v} \mu^d$$

which yields the maximum likelihood estimate (MLE) for μ.

Maximum likelihood estimate of μ under the two-state Markov model

$$\hat{\mu} = d/v$$

5 Statistics and Real-life application

Maximum likelihood estimator of μ under the two-state Markov model $\tilde{\mu} = \mathrm{D}/\mathrm{V}$

As usual we are using capital letters to denote random variables, and lower-case letters to denote sample values.

The following exact results are obtained:

$$E[D_i - \mu V_i] = 0$$
$$\mathrm{var}[D_i - \mu V_i] = E[D_i]$$

Note that the first of these can also be written as $E[D_i] = \mu \cdot E[V_i]$.

5 Statistics and Real-life application

Asymptotic distribution of $\tilde{\mu}$

To find the asymptotic distribution of $\tilde{\mu}$, consider:

$$\frac{1}{N}(D - \mu V) = \frac{1}{N}\sum_{i=1}^{N}(D_i - \mu V_i)$$

We know that $E[D_i - \mu V_i] = 0$ and that $\mathrm{var}[D_i - \mu V_i] = E[D_i]$.

So, by the Central Limit Theorem:

$$\frac{1}{N}(D - \mu V) \sim \mathrm{Normal}\left(0, \frac{E[D]}{N^2}\right)$$

Asymptotically:

$$\tilde{\mu} \sim \mathrm{Normal}\left(\mu, \frac{\mu}{E[V]}\right)$$