INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: M.Sc. SEM 3

Subject : Statistical and Risk Modelling - 3

Chapter: Unit 3 Chapter 2

Chapter Name: The Poisson model

Today's Agenda

1. The Poisson Distribution
2. The Poisson model of mortality
3. Estimating the force of mortality
4. Asymptotic distribution of μ
5. The Poisson Process

1 The Poisson Distribution

The Poisson is a discrete probability distribution in which the random variable can only take non-negative integer values.

A random variable X is said to have a Poisson distribution with mean λ (λ > 0) if the probability function of X is:

$$P(X=x)=\frac{e^{-\lambda}\lambda^{x}}{x!} \qquad \text{for } x=0,1,2,\ldots$$

2 The Poisson model of mortality

If we assume that we observe N individuals as before, for a total of E_χ^c person-years, and that the force of mortality is a constant μ, then a Poisson model is given by the assumption that D has a Poisson distribution with parameter μE_x^c. That is:

$$P(D = d) = \frac{e^{-\mu E_x^c} (\mu E_x^c)^d}{d!}$$

Under the observational plan described above, the Poisson model is not an exact model, since it allows a non-zero probability of more than N deaths, but it is often a very good approximation.

The probability of more than N deaths is usually negligible.

3 Estimating the underlying force of mortality

The Poisson likelihood leads to the following estimator of (constant) μ .

Maximum likelihood estimator of μ under the Poisson model

$$\tilde{\mu} = \frac{D}{E_x^c}$$

Try deriving the above formula for the maximum likelihood estimator of μ .

3 Properties of estimator

The estimator $\tilde{\mu}$ has the following properties:

(i) $E[\tilde{\mu}] = \mu$

So $\tilde{\mu}$ is an unbiased estimator of μ.

(ii) $\mathrm{var}[\tilde{\mu}] = \frac{\mu}{E_x^c}$

In practice, we will substitute $\hat{\mu}$ for μ to estimate these from the data.

4 Asymptotic distribution of $\boldsymbol{\mu}$

When E_x^c is large, the distribution of the estimator $\tilde{\mu}$ is:

$$\tilde{\mu} \sim \text{Normal}\left(\mu, \frac{\mu}{E_x^c}\right)$$

These properties show that this is a sensible estimator to use. Its mean value equals the true value of μ and it varies as little as possible from the true value. The normal approximation allows us to calculate approximate probabilities and confidence intervals for μ .

This model is an approximation to the two-state model and provides the same numerical estimate of μ.

5 The Poisson process

The Poisson process forms the simplest example of a Markov jump process in continuous time.

Definition

The standard time-homogeneous Poisson process is a counting process in continuous time, $\{N_t, t \geq 0\}$, where N_t records the number of occurrences of some type of event within the time interval from 0 to t . The events of interest occur singly and may occur at any time.

The Poisson process is very commonly used to model the occurrence of unpredictable incidents, such as car accidents or arrival of claims at an office.

5 The Poisson process

The probability that an event occurs during the short time interval from time t to time t+h is approximately equal to λh for small h ; the parameter λ is called the rate of the Poisson process.

Formally, an integer-valued process $\{N_t, t \geq 0\}$, with filtration $\{F_t, t \geq 0\}$, is a Poisson process if:

$$P[N_{t+h} - N_t = 1 \mid F_t] = \lambda h + o(h)$$
$$P[N_{t+h} - N_t = 0 \mid F_t] = 1 - \lambda h + o(h)$$
$$P[N_{t+h} - N_t \neq 0,1 \mid F_t] = o(h)$$

where the statement that $f(h) = o(h)$ as $h \to 0$ means $\lim_{h \to 0} \frac{f(h)}{h} = 0$.

As may be seen from the definition, the increment N_{t+h} - N_t of the Poisson process is independent of past values of the process and has a distribution which does not depend on t . It therefore follows that the Poisson process is a process with stationary, independent increments and, in addition, satisfies the Markov property.

5 The Poisson process

Distribution of increments

N_t is a Poisson random variable with mean λt . More generally, N_{t+s} - N_s is a Poisson random variable with mean λt , independent of anything that has occurred before time s .

5 The Poisson process

Sums of independent Poisson processes

Suppose that claims are made to two insurance companies, A and B. The numbers of claims made to each are independent and follow Poisson processes with parameters λA (claims per day) and λB respectively. Then the combined number of claims $(A + B)_t$ is a Poisson process with parameter (λA + λB).

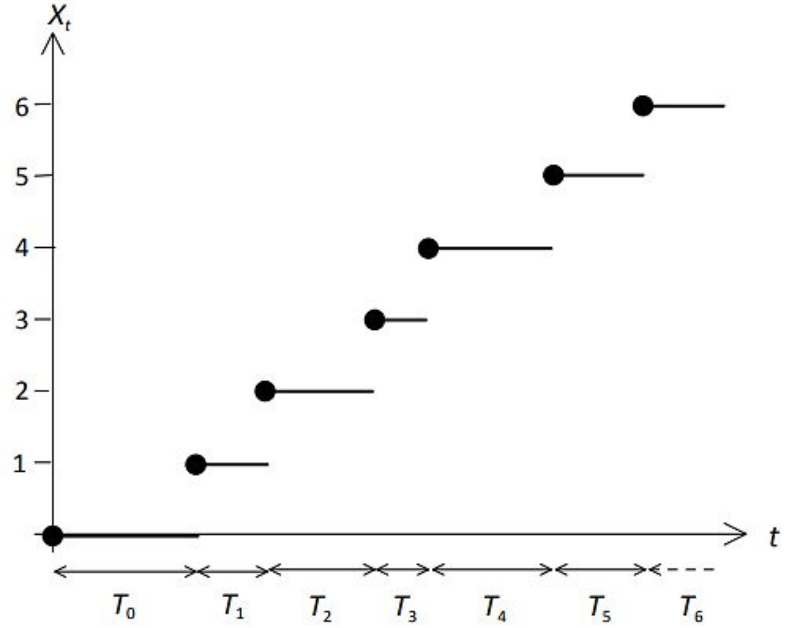
Since the processes are independent of one another, it follows that their increments are independent of one another. These increments are Poisson with parameters λA(t-s) and λB(t-s). Their sum is therefore Poisson, with parameter (λA – λB)(t-s).
They are therefore also stationary and independent.

So, we do have a Poisson process with parameter (λA + λB).

5 The Poisson process

Inter-event times

Since the Poisson process N_t changes only by unit upward jumps, its sample paths are fully characterised by the times at which the jumps take place. Denote by $T_0, T_1, T_2, \ldots$ the successive inter-event times (or holding times), a sequence of random variables.



5 The Poisson process

Distribution of holding time random variables

$T_0, T_1, T_2, \ldots$ is a sequence of independent exponential random variables, each with parameter λ.

Proof:

$P(T_0 > t)$ is the probability that no events occur between time 0 and time t, which is also equal to $P(N_t = 0) = p_0(t) = e^{-\lambda t}$.

Now the distribution function of T_0 is $F(t) = P(T_0 \leq t) = 1 - e^{-\lambda t}, t > 0$, implying that T_0 is exponentially distributed.

Consider now the conditional distribution of T_1 given the value of T_0.

$$\begin{aligned}
P[T_1 > t \mid T_0 = s] \quad &= P[N_{t+s} = 1 \mid T_0 = s] \\
&= P[N_{t+s} - N_s = 0 \mid T_0 = s] \\
&= P[N_{t+s} - N_s = 0] \\
&= p_0(t) \\
&= e^{-\lambda t}
\end{aligned}$$

where the third equality reflects the independence of the increment $N_{t+s} - N_s$ from the past of the process (up to and including time s).