INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Statistical and Risk Modelling - 3

Chapter: Unit 4 Chapter 1 (Part 1)

Chapter Name: Markov process (Time-homogeneous)

Today's Agenda

1. Time-homogeneous Markov Jump Process
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2. The Chapman-Kolmogorov Equations
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1 Time homogeneous Markov Jump Process

Definition - Time-homogeneous Markov jump process are processes in which the transition rates do not vary over time, so the transition probabilities P(X_t = j | X_s = i) depend only on the length of the time interval, t – s.

1.1 Notations

The non-homogeneous (i.e., time-inhomogeneous) Markov model offers particularly rich, and potentially confusing, opportunities to invent different notations for the same quantities. To try to limit any such confusion, we make the following remarks.

We have, $\boldsymbol{p_{ij}(s,t)}$ **to mean the probability of the process being in state $\boldsymbol{j}$ at time $\boldsymbol{t}$, conditional on being in state $\boldsymbol{i}$ at time $\boldsymbol{s \leq t}$.**

The traditional actuarial notation would reserve the symbol t for duration since time s, in which case the above probability would be expressed $p_{ij}(s, s+t)$. Just as likely, the life table symbol ${}_tp_s$ would be adapted, so that $p_{ij}(s, s+t)$ would be written as ${}_tp_s^{ij}$.

We have, $\boldsymbol{\mu_{ij}(s)}$ **to mean the transition rate from state $\boldsymbol{i}$ to state $\boldsymbol{j}$ at time $\boldsymbol{s}$.**
Following the actuarial tradition, the time (or age) may be indicated by a subscript, so that the same rate may be written μ_s^{ij}.

2 The Chapman-Kolmogorov equations

Here we will consider the time-homogeneous case, where probabilities $P(X_t = j \mid X_s = i)$ depend only on the length of the time interval, $t - s$.

The transition probabilities of the Markov jump process:

$$p_{ij}(t) = P(X_t = j \mid X_0 = i)$$

obey the Chapman-Kolmogorov equations:

$$\boldsymbol{p_{ij}(t+s) = \sum_{k \in S} p_{ik}(s) p_{kj}(t) \ \ \textbf{for all } s, t > 0}$$

3 The transition Matrix

Denoting by $P(t)$ the matrix with entries $p_{ij}(t)$, known as the transition matrix, the Chapman Kolmogorov equation reads:

$$P(t+s) = P(s)P(t) \text{ for all } s, t > 0$$

If we know the transition matrix $P(t)$ and the initial probability distribution $q_i = P(X_0 = i)$, we can find general probabilities involving the process X_t by using the Markov property.

For instance, when $0 < t_1 < t_2 < \cdots < t_n$:

$$P\left[x_0 = i, x_{t_1} = j_1, x_{t_2} = j_2, \dots, x_{t_n} = j_n\right] = q_i p_{ij_1}(t_1) p_{j_1 j_2}(t_2 - t_1) \dots p_{j_{n-1} j_n}(t_n - t_{n-1})$$

3.1 Transition Rates

To differentiate the transition probabilities and avoid technical problems with the mathematics, we will make the following assumption.

We will assume that the functions $p_{ij}(t)$ are continuously differentiable.
Noting that:

$$p_{ij}(0) = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

the assumption of differentiability implies the existence of the following quantities:

$$\mu_{ij} = \frac{d}{dt} p_{ij}(t) \bigg|_{t=0} = \lim_{h \to 0} \frac{p_{ij}(h) - \delta_{ij}}{h}$$

$\boldsymbol{\mu_{ij}}$ is the force of transition from state $\boldsymbol{i}$ to state $\boldsymbol{j}$.
Transition rates in time-homogeneous processes do not vary over time.

The function δ_{ij} in the expression above is known as the Kronecker delta.

3.1 Transition Rates

Equivalently, the following relations hold as $h \to 0 (h > 0)$:

$$p_{ij}(h) = \begin{cases} h\mu_{ij} + o(h) & \text{if } i \neq j \\ 1 + h\mu_{ii} + o(h) & \text{if } i = j \end{cases}$$

The interpretation of the first line of is simply that the probability of a transition from i to j during any short time interval $[s, s+h]$ is proportional to h; hence the name transition rate or transition intensity given to μ_{ij}.

3.1 Transition Rates

Generator matrix

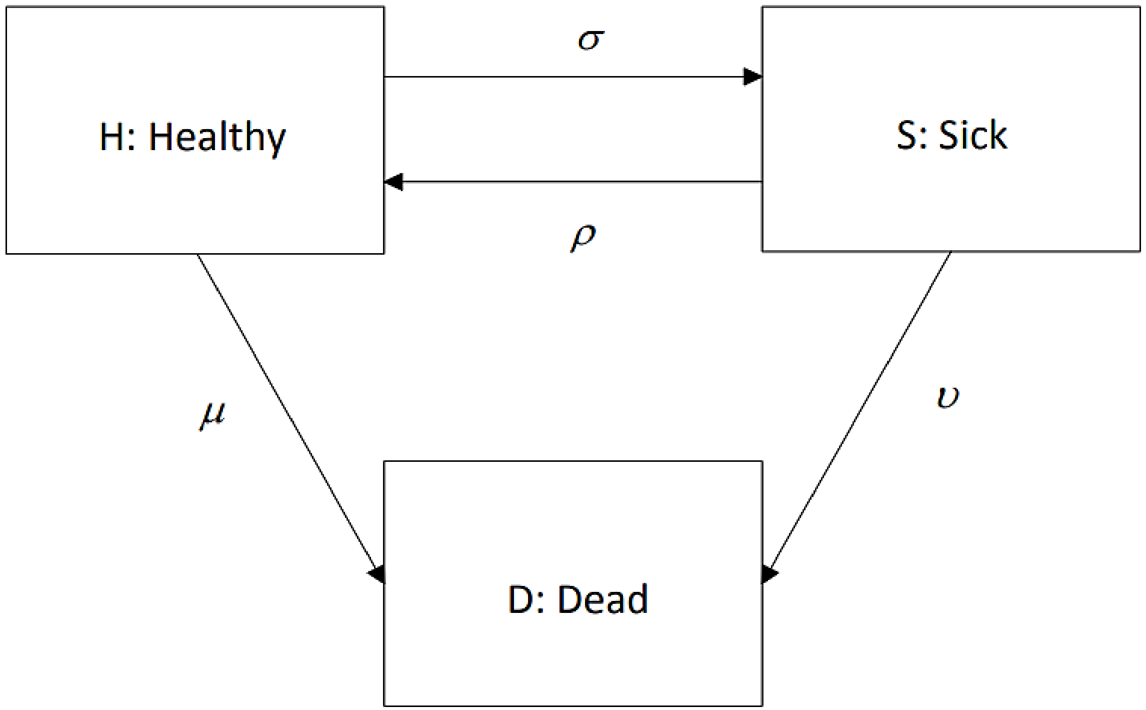
The generator matrix A of a Markov jump process is the matrix of transition rates. In other words, the i, j th entry of A is μ_{ij}.

Hence each row of the matrix A has zero sum.

The relationship $\mu_{ii} = -\sum_{j \neq i} \mu_{ij}$ is often used as a working definition of μ_{ii}.

The transition rate μ_{ii} is then defined as minus the sum of the transition rates out of state i.

4 The time-homogeneous health-sickness-death model



What will be the transition matrix for the above diagram?

4 The time-homogeneous health-sickness-death model

The generator matrix for the HSD model is:

$$A = \begin{pmatrix} -\sigma-\mu & \sigma & \mu \\ \rho & -\rho-v & v \\ 0 & 0 & 0 \end{pmatrix}$$

Here the order of the states has been taken to be H, S, then D (as usual).

5.1 Kolmogorov's forward differential equations

Transition rates are of fundamental importance in that they characterize fully the distribution of Markov jump processes. To see this, substitute $t = h$ and $s = t$ in Chapman eqn:

$$p_{ij}(t+h) = \sum_{k \in S} p_{ik}(t) p_{kj}(h) = p_{ij}(t) + h \sum_{k \in S} p_{ik}(t) \mu_{kj} + o(h)$$

The second equality follows from the relationship:

$$p_{kj}(h) = \begin{cases} h\mu_{kj} + o(h) & \text{if } j \neq k \\ 1 + h\mu_{kk} + o(h) & \text{if } j = k \end{cases}$$

This leads to the differential equation:

$$\frac{d}{dt} p_{ij}(t) = \sum_{k \in S} p_{ik}(t) \mu_{kj} \quad \textbf{for all } i, j$$

5.1 Kolmogorov's forward differential equations

Kolmogorov's forward differential equations (time-homogeneous case) These can be written in compact (i.e., matrix) form as:

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\boldsymbol{P}(\boldsymbol{t}) = \boldsymbol{P}(\boldsymbol{t})\boldsymbol{A}$$

where A is the matrix with entries μ_{kj}.

Recall that A is often called the generator matrix of the Markov jump process.

5.2 Kolmogorov’s backward differential equations

Kolmogorov's backward differential equations (time-homogeneous case)

These can be written in matrix form as:

$$\frac{d}{dt}P(t) = AP(t)$$

5.3 Kolmogorov's differential equations

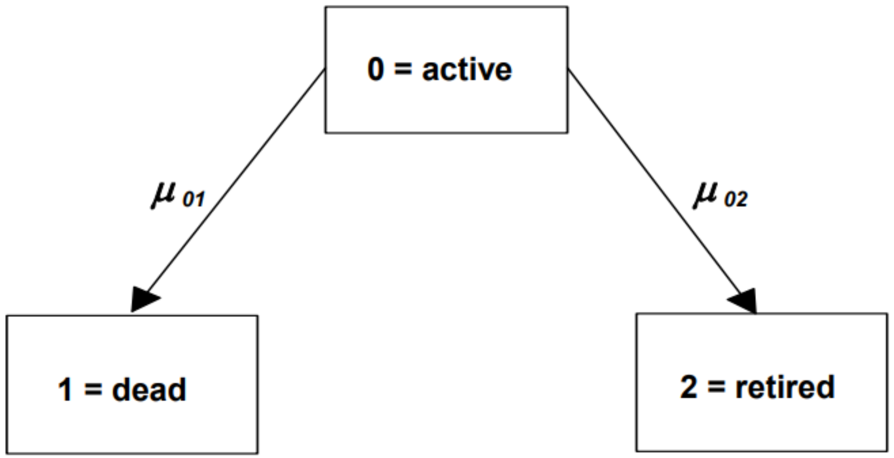
Elementary Cases

For example, consider the two-decrement model, in which the transition intensities are constant.

The Solution to the Kolmogorov equation can be given as:

$$p_{01}(x, x+t) = \frac{\mu_{01}}{\mu_{01} + \mu_{02}}\left[1 - e^{-(\mu_{01}+\mu_{02})t}\right]$$

$$p_{02}(x, x+t) = \frac{\mu_{02}}{\mu_{01} + \mu_{02}}\left[1 - e^{-(\mu_{01}+\mu_{02})t}\right]$$



6 Poisson Process

Holding times and occupancy probabilities

Distribution of the first holding time

The first holding time of a time-homogeneous Markov jump process with transition rates μ_{ij} is exponentially distributed with parameter:

$$\lambda_i = -\mu_{ii} = \sum_{j \neq i} \mu_{ij}$$

In other words:

$$P\left[T_0 > t \mid X_0 = i\right] = e^{-\lambda_i t}$$

6 Poisson Process

Holding times and occupancy probabilities

Probability that the process goes into state j when it leaves state I

Given that a time-homogeneous Markov jump process is currently in state i , the probability that it moves into state j when it leaves state i is given by:

$$\frac{\mu_{ij}}{\lambda_i} = \frac{\text{the force of transition from state } i \text{ to state } j}{\text{the total force of transition out of state } i}$$

Also, given a jump has occurred, the time at which it took place does not affect the probability of the jump being to a particular state.

6 Poisson Process

Holding times and occupancy probabilities

Distribution of holding time random variables and occupancy probabilities

For a time-homogeneous Markov jump process, let W_i denote the holding time (or waiting time) in state i. Then:

$$W_i \sim Exp(\lambda_i)$$

where λ_i is the total force of transition out of state i.

So the probability of staying in state i for at least t time units (*ie* the occupancy probability for state i) is:

$$P(W_i > t) = p_{\overline{ii}}(t) = e^{-\lambda_i t}$$

6 Expected time to reach state k starting from state i

Let m_i denote the expected time for the process to reach state k given that it is currently in state i. Then m_i can be calculated using the recursive formula:

$$m_i = \frac{1}{\lambda_i} + \sum_{j \neq i,k} \frac{\mu_{ij}}{\lambda_i} m_j$$

This formula is given on page 38 of the *Tables*. Note that the *Tables* use the notation σ_{ij} instead of μ_{ij} to denote the force of transition from state i to state j.

7 The Jump Chain

If a Markov jump process is examined only at the times of its transitions, the resulting process, denoted $\{\hat{X}_n : n = 0,1,\ldots\}$, where $\hat{X}_0$ is the initial state, and for $n \geq 1$:

$$\hat{X}_n = X_{T_0 + T_1 + \cdots + T_{n-1}}$$

is called the *jump chain* associated with X.

The only way in which the jump chain differs from a standard Markov chain is when the jump process $\{X_t$, t≥ 0$\}$ encounters an absorbing state. From that time on it makes no further transitions, implying that time stops for the jump chain.

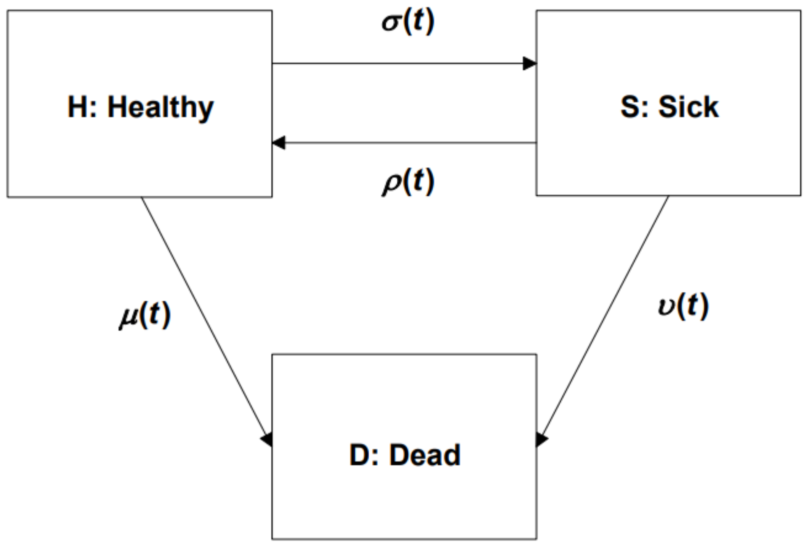
To deal with the jump chain entirely within the framework of Markov chains it is permissible to treat the absorbing state in the same way as for a Markov chain, so that transitions continue to occur, but the chain remains in the same state after the transition.

8 Maximum Likelihood Estimators

Consider the illness-death model, which has three states: healthy (H), sick (S) and dead (D):

The observations in respect of a single life are now: (a) the times between successive transitions; and (b) the numbers of transitions of each type.

If the transition intensities are constant, each spell of length t in the healthy or sick states contributes a factor of the form $e^{-(\mu+\sigma)t}$ or $e^{-(v+\rho)t}$ respectively to the likelihood, so it suffices to record the total waiting time spent in each state.



8 Maximum Likelihood Estimators

Define:

V_i = Waiting time of the i th life in the healthy state

W_i = Waiting time of the i th life in the sick state

S_i = Number of transitions healthy $\to$ sick by the ith life

R_i = Number of transitions sick $\to$ healthy by the ith life

D_i = Number of transitions healthy $\to$ dead by the ith life

U_i = Number of transitions sick $\to$ dead by the ith life

We also need to define totals $V = \sum_{i=1}^{N} V_i$ (and so on).

8 Maximum Likelihood Estimators

Using lower case symbols for the observed samples as usual, it is easily shown that the likelihood for the four parameters, μ, σ, v, ρ, given the data is proportional to:

$$L(\mu,\upsilon,\sigma,\rho) = e^{-(\mu+\sigma)v} e^{-(\upsilon+\rho)w} \mu^{d} \upsilon^{u} \sigma^{s} \rho^{r}$$

Solving the above likelihood, gives the estimators as:

The maximum likelihood estimators are:

$$\tilde{\mu} = \frac{D}{V}, \quad \tilde{\upsilon} = \frac{U}{W}, \quad \tilde{\sigma} = \frac{S}{V}, \quad \tilde{\rho} = \frac{R}{W}$$

8 Maximum Likelihood Estimators

Estimating transition rates in a time-homogeneous Markov jump process

The maximum likelihood estimate of the transition rate μ_{ij} is:

$$\hat{\mu}_{ij} = \frac{n_{ij}}{t_i}$$

where n_{ij} is the number of transitions from state i to state j, and t_i is the total waiting time (or total holding time) in state i.

8 Maximum Likelihood Estimators

The maximum likelihood estimator of μ_{ij} has the following properties:

- it is asymptotically normally distributed
- it is asymptotically unbiased
- asymptotically, its variance is given by the Cramér-Rao lower bound (CRLB).

  The formula for the CRLB is given on page 23 of the Tables.