INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject : Statistical & Risk Modelling -3

Time Inhomogeneous Markov Jump

Agenda

1. Time In-homogenous Markov jump process
2. Chapman Kolmogorov equation
3. Kolmogorov Forward differential equations
4. Kolmogorov ~~Forward~~ Backward differential equations
5. Occupancy probability
6. Probability of going to another state
7. Integrated form of KFD and KBD
8. Applications

Time Inhomogeneity

In this chapter we discuss time-inhomogeneous Markov jump processes. The transition probabilities $P(X_t = j \mid X_s = i)$ for a time-inhomogeneous process depend not only on the length of the time interval $[s,t]$, but also on the times s and t when it starts and ends. This is because the transition rates for a time-inhomogeneous process vary over time.

$P_{HS}(5) \leftarrow$ duration based probability $\rightarrow$ Time homogeneous

$P_{HS}(20, 25) \neq P_{HS}(60, 65) \rightarrow$ time and duration based prob. $\rightarrow$ Time inhomogeneous

i.e. $P_{ij}(s, t)$

Chapman Kolmogorov Eqns

The more general continuous-time Markov jump process $\{X_t, t \geq 0\}$ has transition probabilities:

$$p_{ij}(s,t) = P[X_t = j | X_s = i] \quad (s \leq t)$$

which obey a version of the *Chapman-Kolmogorov equations*, written in matrix form as:

$$P(s,t) = P(s,u)P(u,t) \qquad \text{for all } s < u < t$$

or equivalently:

$$p_{ij}(s,t) = \sum_{k \in S} p_{ik}(s,u)\, p_{kj}(u,t) \qquad \text{for all } s < u < t$$

Time homogen.

$$P_{ij}(t+h) = \sum_{k \in S} P_{ik}(t) \cdot P_{kj}(h)$$

i . . . . k . . . . j

s . . . . u . . . . t

$$\boxed{P_{ij}(s,t) = \sum_{k \in S} P_{ik}(s,u) \cdot P_{kj}(u,t)}$$

Kolmogorov Eqns updated

Kolmogorov's forward differential equations (time-inhomogeneous case)

Written in matrix form these are:

$$\frac{\partial}{\partial t}P(s,t) = P(s,t)A(t)$$

where $A(t)$ is the matrix with entries $\mu_{ij}(t)$.

$$\frac{dP_{ij}(s,t)}{dt} = \sum_{k \in S.S.} P_{ik}(s,t) \cdot \mu_{kj}(t)$$

Kolmogorov's backward differential equations (time-inhomogeneous case)

The matrix form of Kolmogorov's backward equations is:

$$\frac{\partial}{\partial s}P(s,t) = -A(s)P(s,t)$$

$$\frac{dP_{ij}(s,t)}{dt} = -\sum_{k \in S.S.} P_{kj}(s,t) \cdot \mu_{ik}(s)$$

Prove using Chapman Kolmogorov equation

JAQS

Kolmogorov Eqns updated

$\mu_{ii} = -\sum_{i \neq j} \mu_{ij}$

$\mu_{ss}(t) = -\mu_{SH}(t) - \mu_{SD}(t)$

H/S → D

Question

Write down Kolmogorov's forward and backward differential equation for $p_{HD}(s,t)$ & $p_{Hs}(s,t)$.

KFD | KBD

$$\frac{d\,p_{HD}(s,t)}{dt} = p_{HH}(s,t)\cdot\mu_{HD}(t) + p_{HS}(s,t)\cdot\mu_{SD}(t) \quad\Big|\quad -\left[p_{HD}(s,t)\cdot\mu_{HH}(s) + p_{SD}(s,t)\cdot\mu_{HS}(s) + p_{DD}(s,t)\,\mu_{HD}(s)\right]$$

$$\frac{d\,p_{HS}(s,t)}{dt} = p_{HH}(s,t)\cdot\mu_{HS}(t) + p_{HS}(s,t)\cdot\mu_{SS}(t) \quad\Big|\quad -\left[p_{HS}(s,t)\cdot\mu_{HH}(s) + p_{SS}(s,t)\cdot\mu_{HS}(s)\right]$$

Other probabilities

Prob. of staying contiguously in the same state

Occupancy probabilities for time-inhomogeneous Markov jump processes

For a time-inhomogeneous Markov jump process:

$$p_{\overline{ii}}(s,t) = \exp\left(-\int_s^t \lambda_i(u)\,du\right) = \exp\left(-\int_0^{t-s} \lambda_i(s+u)\,du\right)$$

where $\lambda_i(u)$ denotes the total force of transition out of state i at time u.

$p_{\overline{ii}}(s,t) =$

$\exp\left[-\int_s^t \lambda_i(u)\cdot du\right]$

Prove using Chapman Kolmogorov equation

$\lambda_1 \lambda_2 \lambda_3 \ldots\ldots$

sum → Total intensity

→ $\lambda t \leftarrow \lambda$ is const

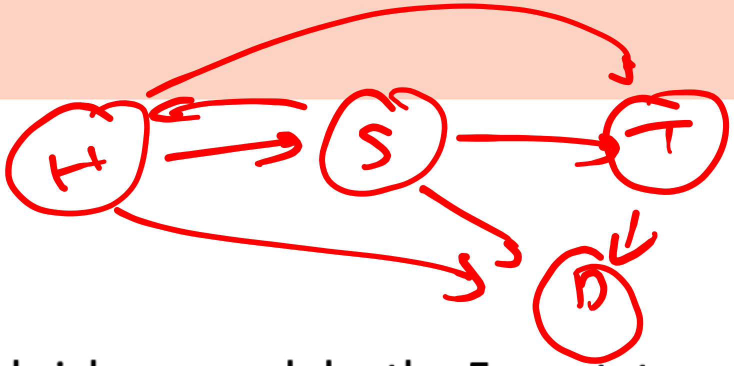
$\int_0^t \lambda(s)\cdot ds$

$\boxed{P_{\overline{ii}}(s,t)} \neq P_{ii}(s,t)$

$P_{\overline{ii}}(t) = e^{-\lambda_i t}$

λ_i was constant

Other probabilities



$$\cancel{\frac{d\,p_{TD}(50,51)}{dt} = p_{TT}(50,51)\cdot\mu(51)}$$

Question

A Markov jump process is used to model sickness and death. Four states are included, namely H, S_1, S_2 and D, which represent healthy, sick, terminally sick and dead, respectively. We are told that the people who are terminally sick never recover and die at a rate of $1.03(1.01)^t$ where t is their age in years.

$$\mu_{TD}(t) = 1.03(1.01)^t$$

Calculate the probability that a terminally sick 50-year-old dies within a year.

$$p_{TD}(50,51) = ?$$

$$p_{TD}(50,51) = 1 - p_{TT}(50,51)$$

$$= 1 - p_{\overline{TT}}(50,51)$$

$$p_{TT}(50,51) = p_{\overline{TT}}(50,51)$$

Other probabilities

Question

$$p_{TD}(50,51) = 1 - \exp\left[-\int_{50}^{51} \mu_{TD}(u)\,.du\right]$$

$$= 1 - \exp\left[-\int_{50}^{51} 1.03\,(1.01)^{u}\,.du\right]$$

$$= 1 - \exp\left[-\left[\frac{1.03\,(1.01)^{u}}{\ln 1.01}\right]_{50}^{51}\right]$$

$$= 1 - \exp\left[-\left(\frac{1.03\times 1.01^{51} - 1.03\times 1.01^{50}}{\ln 1.01}\right)\right]$$

$$= 0.8177$$

JAQS

Other probabilities

i

s --- w --- s+w → jump

→ Residual holding time

Probability that the process goes into state *j* when it leaves state *i*

Given that the process is in state i at time s and it stays there until time $s+w$, the probability that it moves into state j when it leaves state i at time $s+w$ is:

$$\frac{\mu_{ij}(s+w)}{\lambda_i(s+w)} = \frac{\text{the force of transition from state } i \text{ to state } j \text{ at time } s+w}{\text{the total force out of state } i \text{ at time } s+w}$$

Time homogen. → Prob. (go i → to state j | leave i)

$$= \frac{\mu_{ij}}{\lambda_i} \rightarrow \lambda_i = \sum_{i \neq j} \mu_{ij}$$

Time inhomogen → $= \boxed{\dfrac{\mu_{ij}(s+w)}{\lambda_i(s+w)}}$

Residual Holding time

→ time I will be in the same state

$$p_{\overline{ii}}(s,t) = \exp\left(-\int_s^t \lambda_i(u)\,du\right) = \exp\left(-\int_0^{t-s} \lambda_i(s+u)\,du\right)$$

P[will remain in state 'i' for at least time w]

For a general Markov jump process, $\{X_t, t \geq 0\}$, define the *residual holding time* R_s as the (random) amount of time between s and the next jump:

$$\{R_s > w, X_s = i\} = \{X_u = i, s \leq u \leq s+w\}$$

$$P[R_s > w \mid X_s = i] = \exp\left(-\int_s^{s+w} \lambda_i(t)\,dt\right)$$

i ... i

s ↓ $s+w$

Prob = ?

$P[R_s > w / X_s = i]$

Residual holding time

$= p_{\overline{ii}}(s, s+w)$

$= \exp\left[-\int_s^{s+w} \lambda_i(u)\,du\right]$

Residual Holding time

$$P[R_s > \omega \mid X_s = i] = \exp\left[-\int_s^{s+\omega} \lambda_i(u)\, du\right]$$

$$\therefore \underbrace{P[R_s \le \omega \mid X_s = i]}_{\text{cdf of } R_s} = \underline{1 - \exp\left[-\int_s^{s+\omega} \lambda_i(u)\, du\right]}$$

Pdf $\rightarrow$ calculated by diff. cdf wrt ω

$$\therefore \text{PDF} = \frac{d\left[1 - \exp\left(-\int_s^{s+\omega} \lambda_i(u)\, du\right)\right]}{d\omega}$$

Residual Holding time

exp + $\underline{\lambda e^{-\lambda t}}$ $\Lambda \to$ capital lamda

$$= \frac{d\left[1 - \exp\left(-\left(\Lambda(s+\omega) - \Lambda(s)\right)\right)\right]}{d\omega}$$

$$\int \lambda \cdot du = \Lambda(u)$$

$$= -\exp\left[-\left(\Lambda(s+\omega) - \Lambda(s)\right)\right] \cdot \frac{d\left(-\Lambda(s+\omega)\right)}{d\omega}$$

$$= \exp\left[-\left(\Lambda(s+\omega) - \Lambda(s)\right)\right] \cdot \lambda(s+\omega)$$

$$= \underline{\lambda(s+\omega)\, \exp\left[-\int_s^{s+\omega} \lambda_i(u) \cdot du\right]}$$

Current Holding time → time I have spent in the current state

For a full justification of this equation one needs to appeal to the properties of the *current holding time* C_t , namely the time between the last jump and t :

$$\{C_t \geq w, X_t = j\} = \{X_u = j, t - w \leq u \leq t\}$$

$$\downarrow$$

$$P[C_t > w \mid X_t = j]$$

$$\downarrow$$

$$P_{jj}(t-w, t)$$

last jump   i R_s next jump

$s-w$ s $s+w$

C_t

jump to state i

$t-w$ t

$C_t \leftarrow$ Current holding time

All Prob. cal^n of C_t are exactly the same as R_s, only diff. be:

J → start time → $t-w$

end time → t

Integrated form of Kolmogorov Backward Eqn

$$p_{ij}(s,t)=\sum_{l\neq i}\int_{0}^{t-s} e^{-\int_s^{s+w}\lambda_i(u)du}\,\mu_{il}(s+w)\,p_{lj}(s+w,t)\,dw$$

provided $l \neq i$.

1 • the probability of remaining in state i from time s to time $s+w$

2 • then making a transition to state l at time $s+w$

3 • and finally going from state l at time $s+w$ to state j at time t.

Integrated form of Kolmogorov Forward Eqn

$$p_{ij}(s,t) = \sum_{k \neq j} \int_0^{t-s} p_{ik}(s,t-w)\,\mu_{kj}(t-w)\,p_{\overline{jj}}(t-w,t)\,dw$$

for $j \neq \cancel{i}\,k$

The factors in the integral are:

- the probability of going from state i at time s to state k at time $t-w$
- then making a transition from state k to state j at time $t-w$
- and staying in state j from time $t-w$ to time t.

Exam style question

(i) State the condition needed for a Markov Jump Process to be time inhomogeneous. [1]

→ when transition rates are dependent on time, the process is time-inhomogen.

(ii) Describe the ~~principal~~ Practical difficulties in modelling using a Markov Jump Process with time inhomogeneous rates. [2]

→ Estimating the transition rate is difficult
→ Data requirements are very high.
→ Solving kolmogorov eq^ns is not straightforward.

A multi-tasking worker at a children's nursery observes whether children are being 'Good' or 'Naughty' at all times. Her observations suggest that the probability of a child moving between the two states varies with the time, t, since the child arrived at the nursery in the morning. She estimates that the transition rates are:

From Good to Naughty: $0.2+0.04t$

From Naughty to Good: $0.4-0.04t$

where t is measured in hours from the time the child arrived in the morning, $0 \le t \le 8$.

Exam style question

A child is in the 'Good' state when he arrives at the nursery at 9am.

(iii) Calculate the probability that the child is Good for all the time up until time t. [3]

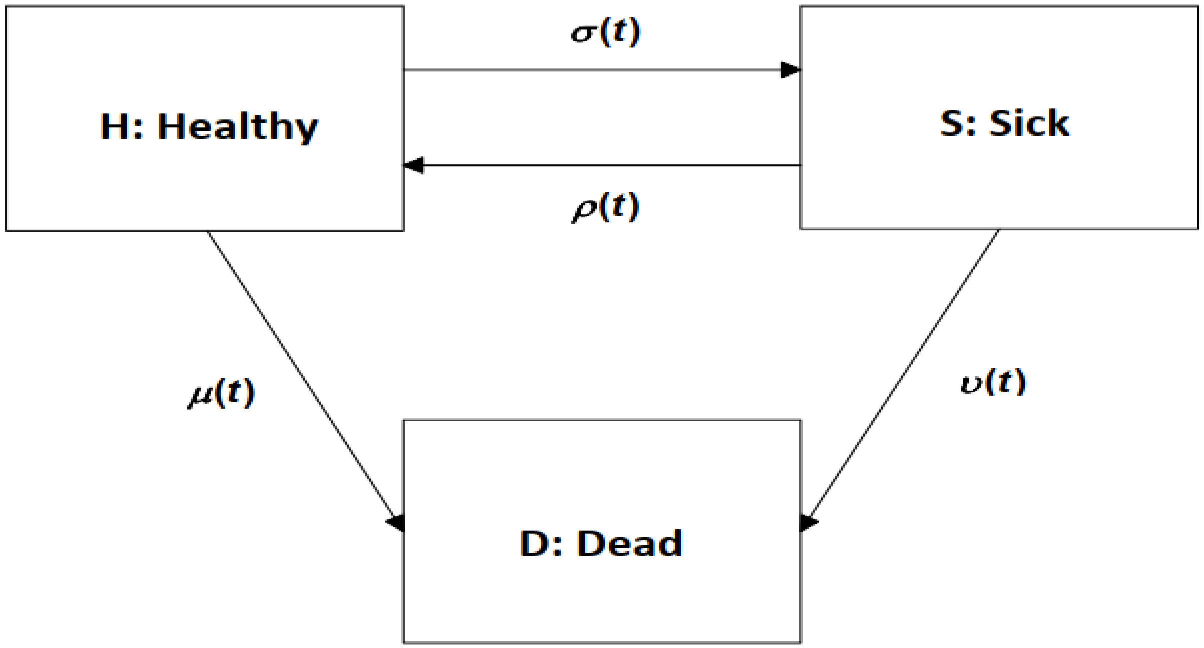
(iv) Calculate the time by which there is at least a 50% chance of the child having been Naughty at some point. [2]

Let $P_G(t)$ be the probability that the child is Good at time t.

(v) Derive a differential equation just involving $P_G(t)$ which could be used to determine the probability that the child is Good on leaving the nursery at 5pm. [2]

$\frac{d P_{GG}(0,t)}{dt}$:

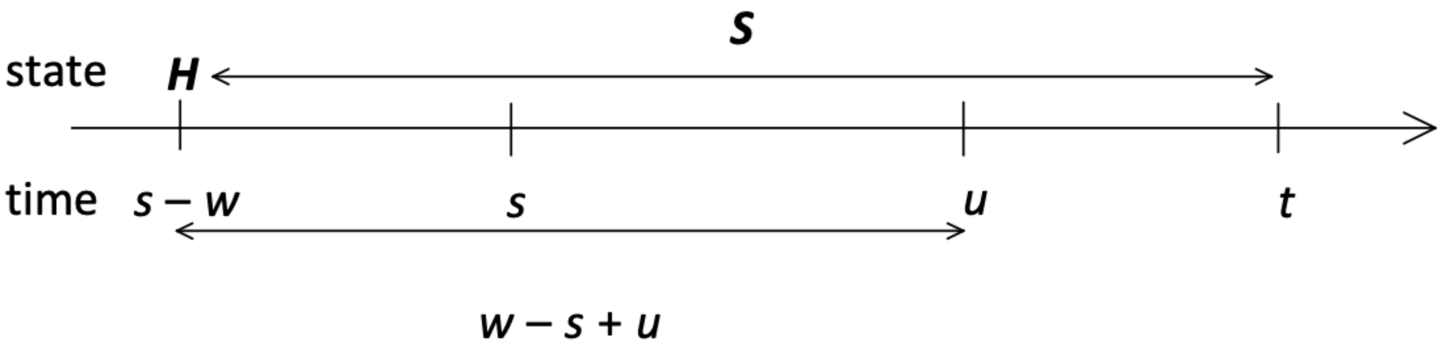
Sickness and Death Model



$$A(t) = \begin{pmatrix} -\sigma(t) - \mu(t) & \sigma(t) & \mu(t) \\ \rho(t) & -\rho(t) - \upsilon(t) & \upsilon(t) \\ 0 & 0 & 0 \end{pmatrix}$$

Sickness and Death with Duration

To calculate the probability of remaining continuously sick during [s,t] given a current illness period [s – w , s], one needs to update the values of ρ and ν as the illness progresses



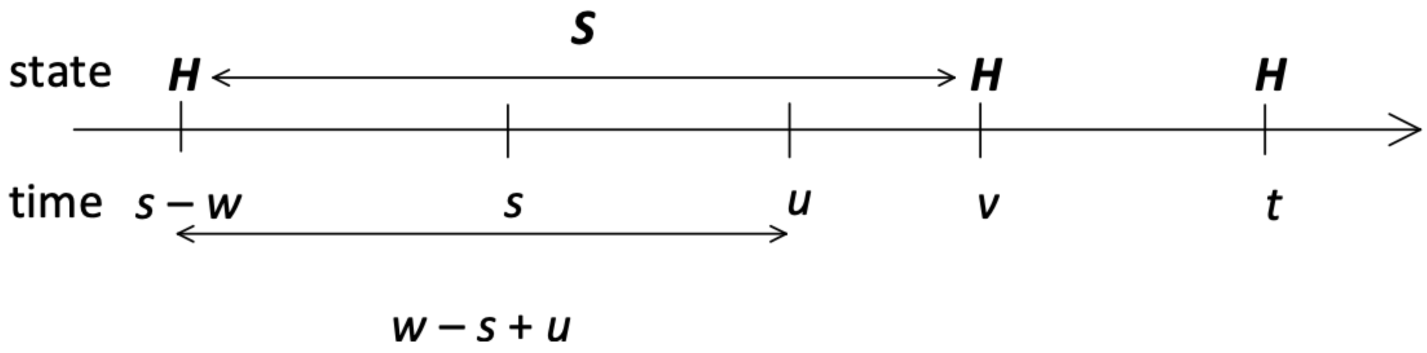
$$P\left[X_t = S, R_s > t - s \mid X_s = S, C_s = w\right] = \exp\left[-\int_s^t \left(\rho(u, w-s+u) + \upsilon(u, w-s+u)\right) du\right]$$

Sickness and Death with Duration

As a final example, the probability of being healthy at time t given that you are sick at time s with current illness duration w can be written as:

$$p_{S_wH}(s,t) = P\left[X_t = H \mid X_s = S, C_s = w\right]$$

$$= \int_s^t e^{-\int_s^v \left(\rho(u,w-s+u)+\nu(u,w-s+u)\right)du}\,\rho(v,w-s+v)\,p_{HH}(v,t)\,dv$$



Sickness and Death with Duration

Consider again the marriage model above ., only now assume that the transition rate $d(t)$ depends on the current holding time. (So the chance of divorce depends on how long a person has been married.) Write down expressions for the probability that:

(i) a bachelor remains a bachelor throughout a period $[s,t]$

(ii) a person who gets married at time $s-w$ and remains married throughout $[s-w,s]$, continues to be married throughout $[s,t]$

(iii) a person is married at time t and has been so for at least time w, given that they were divorced at time $s<t-w$.

Thank You