Small time transition prob.

$$\mu_{ii} = -\sum_{i \neq j} \mu_{ij} = -\lambda_i$$

$$\lim_{h \to 0} P_{ij}(t, t+h) = h \cdot \mu_{ij}(t) + o(h)$$

$$= h \cdot \mu_{ij}(t) + o(h), \quad i \neq j$$

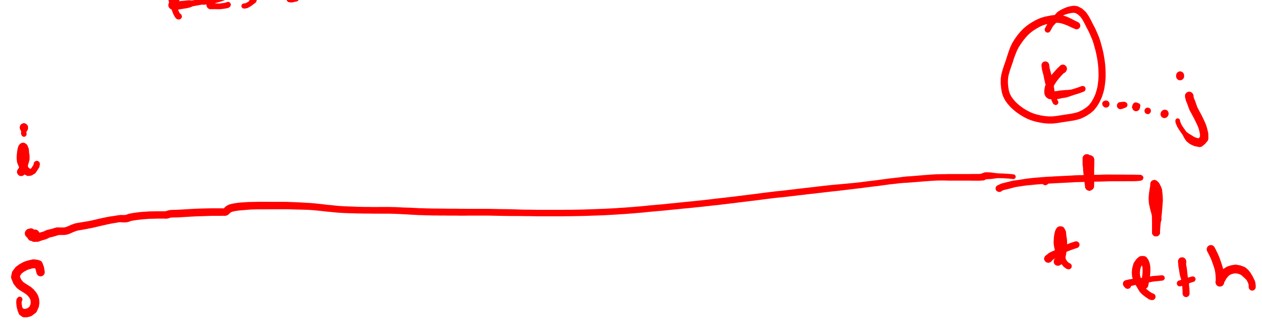
$$= 1 + h \cdot \mu_{ii}(t) + o(h), \quad i = j$$

$$= 1 - h \sum_{i \neq j} \mu_{ij}(t) + o(h)$$

KFD :

[KFD is formed to create a relation b/w transition intensities and finite time transition probabilities]

$$p_{ij}(s, t+h) = \sum_{k \in S.S.} p_{ik}(s, t) \cdot \underbrace{p_{kj}(t, t+h)} \rightarrow [\text{C.K.E.}]$$



$k \neq j \quad \longleftarrow \quad \longrightarrow \quad k = j$

$\boxed{\mu_{ii} = -\sum_{i \neq j} \mu_{ij}}$

$$P_{ij}(s, t+h) =$$

$$= \sum_{k \neq j} P_{ik}(s,t) \cdot \underbrace{P_{kj}(t, t+h)} + P_{ij}(s,t) \cdot \underbrace{P_{jj}(t, t+h)}$$

$$= \sum_{k \neq j} P_{ik}(s,t) \left[h \cdot \mu_{kj}(t) + o(h) \right] + P_{ij}(s,t) \left[1 + h \cdot \mu_{jj}(t) + o(h) \right]$$

$$P_{ij}(s, t+h) = \sum_{k \neq j} \left[P_{ik}(s,t) \cdot h \cdot \mu_{kj}(t) + o(h) \cdot P_{ik}(s,t) \right] + \underline{P_{ij}(s,t)}, + h \cdot \mu_{jj}(t) \cdot P_{ij}(s,t) + o(h) \cdot P_{ij}(s,t)$$

$$\frac{p_{ij}(s,t+h) - p_{ij}(s,t)}{h} = \mu_{jj}(t)\cdot p_{ij}(s,t) + \frac{o(h)}{h}\cdot p_{ij}(s,t)$$

$$+ \sum_{k\neq j}\left[\mu_{kj}(t)\cdot p_{ik}(s,t) + \frac{o(h)}{h}\cdot p_{ik}(s,t)\right]$$

$$\lim_{h\to 0} \frac{p_{ij}(s,t+h) - p_{ij}(s,t)}{h} = \mu_{jj}(t)\cdot p_{ij}(s,t)$$

$$+ \sum_{k\neq j}\left[\mu_{kj}(t)\cdot p_{ik}(s,t)\right]$$

$$\boxed{\frac{d\, p_{ij}(s,t)}{dt} = \sum_{k\in S.S.} p_{ik}(s,t)\cdot \mu_{kj}(t)}$$

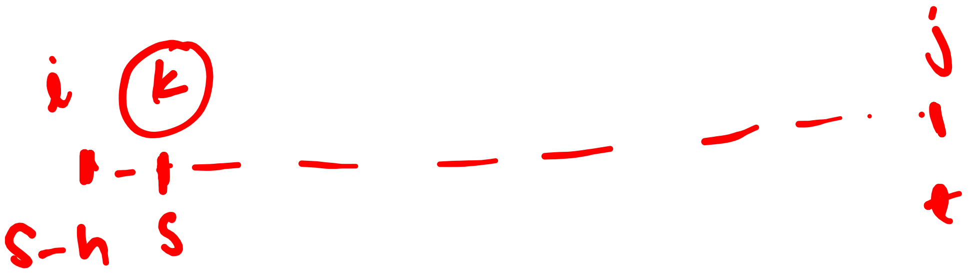
KBD:

To Prove:

$$\boxed{\frac{d\,P_{ij}(s,t)}{dt} = -\sum_{k \in S} \mu_{ik}(s) \cdot P_{kj}(s,t)}$$

C.K.E.

$$P_{ij}(s-h,t) = \sum_{k \in S} P_{ik}(s-h,s) \cdot P_{kj}(s,t)$$



$\boxed{\text{5-6 mins}}$

$$P_{ij}(s-h,t) = \sum_{k \in S.S} P_{ik}(s-h,s) \cdot P_{kj}(s,t)$$

$i = k$ $\qquad\qquad$ $i \neq k$

$$= \underbrace{P_{ii}(s-h,s)} \cdot P_{ij}(s,t) + \sum_{i \neq k} \underbrace{P_{ik}(s-h,s)} \cdot P_{kj}(s,t)$$

$$= \left[1 + h \cdot \mu_{ii}(s-h) + o(h)\right] \cdot P_{ij}(s,t) + \sum_{i \neq k} P_{kj}(s,t) \cdot \left[h \cdot \mu_{ik}(s-h) + o(h)\right]$$

$$= P_{ij}(s,t) + h \cdot \mu_{ii}(s-h) \cdot P_{ij}(s,t) + o(h) \cdot P_{ij}(s,t) + \sum_{i \neq k} \left[P_{kj}(s,t) \cdot h \cdot \mu_{ik}(s-h) + o(h) \cdot P_{kj}(s,t)\right]$$

$$\frac{P_{ij}(s-h,t) - P_{ij}(s,t)}{h} = P_{ij}(s,t)\cdot \mu_{ii}(s-h) + \frac{o(h)}{h}\cdot P_{ij}(s,t) + \sum_{i\neq k}\left[\mu_{ik}(s-h)\cdot P_{kj}(s,t) + \frac{o(h)}{h} P_{kj}(s,t)\right]$$

$$\lim_{h\to 0} \frac{P_{ij}(s-h,t) - P_{ij}(s,t)}{h} = P_{ij}(s,t)\cdot \mu_{ii}(s) + \sum_{i\neq k} \mu_{ik}(s)\cdot P_{kj}(s,t)$$

$$-\lim_{h\to 0} \frac{P_{ij}(s,t) - P_{ij}(s-h,t)}{h} = \sum_{k\in S.S.} \mu_{ik}(s)\cdot P_{kj}(s,t)$$

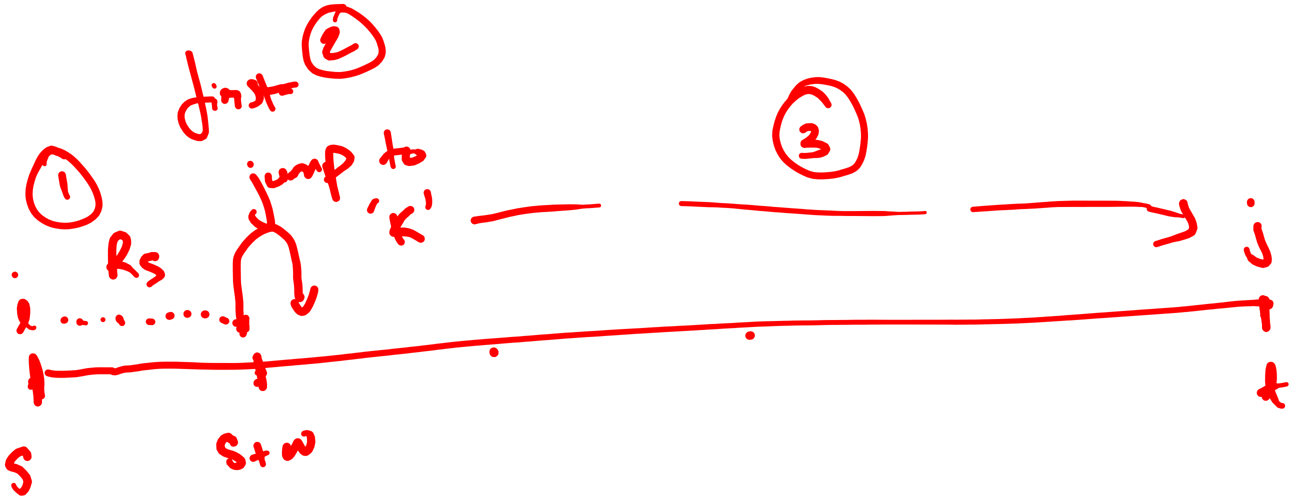
$$\boxed{\frac{d\, P_{ij}(s,t)}{dt} = -\sum_{k\in S.S.} \mu_{ik}(s)\cdot P_{kj}(s,t)}$$

Int. KBE

$$\exp\left[-\int_s^{s+w}\lambda_i(u)\,du\right]$$

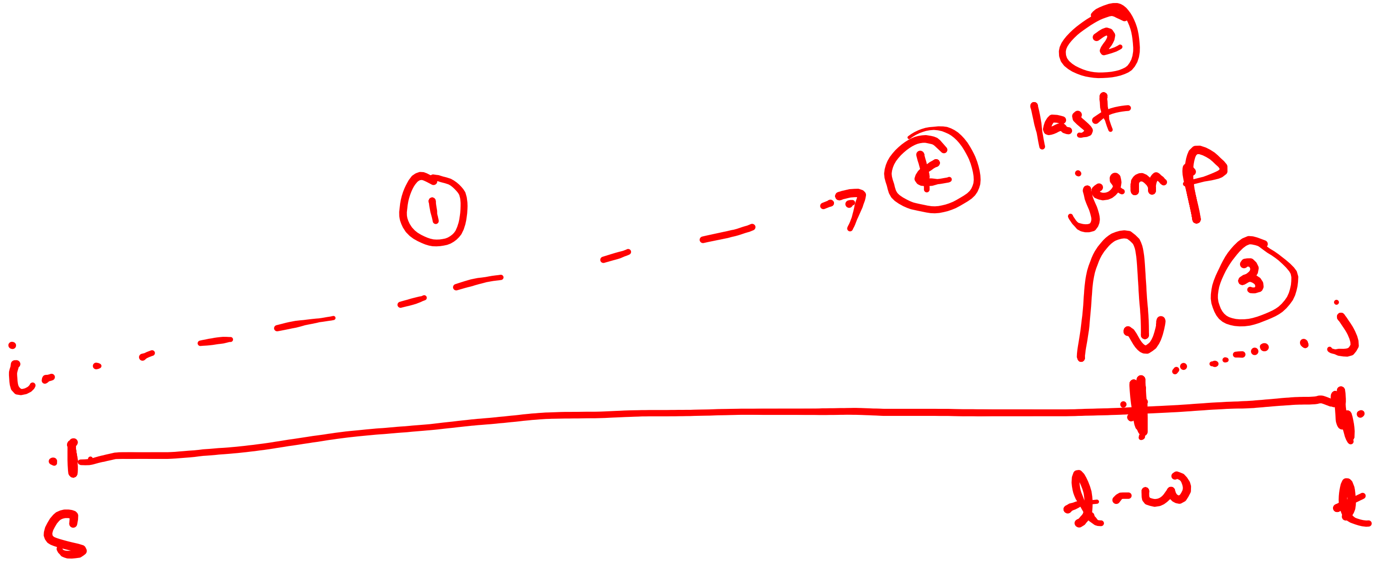
$$P_{ij}(s,t) = \sum_{k\neq i}\int_0^{t-s} \overset{①}{P_{\bar{i}\bar{i}}(s,\, s+w)}\cdot \overset{②}{\mu_{ik}(s+w)}\cdot \overset{③}{\underline{P_{kj}(s+w,\, t)}}\cdot dw$$

(Here, 'k' and 'j' could be the same)



Int. KFE

$$P_{ij}(s,t) = \sum_{k \neq j} \int_0^{t-s} \overset{1}{P_{ik}(s, t-w)} \cdot \overset{2}{\mu_{kj}(t-w)} \cdot \overset{3}{P_{\overline{jj}}(t-w, t)} \, dw$$



Sol:-

$$P_{\bar{i}\bar{i}}(s,t) = \exp\left[-\int_s^t \lambda_i(u)\,du\right]$$

$G \to N \to 0.2 + 0.04t$

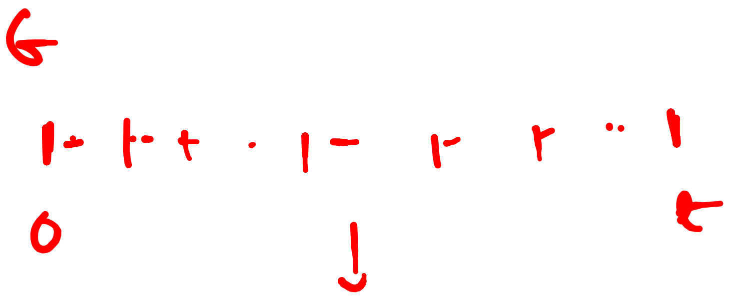
$N \to G \to 0.4 - 0.04t$

iii) Prob. that child is Good till time 't'.

$$P_{\bar{G}G}(0,t) = \exp\left[-\int_0^t \lambda_G(u)\,du\right]$$

$$= \exp\left[-\int_0^t (0.2 + 0.04s)\,ds\right]$$

$$= \exp\left[-\left(0.2s + \frac{0.04s^2}{2}\right)_0^t\right] = \exp\left[-(0.2t + 0.02t^2)\right]$$



Either child has been naughty at some pt. or that he has never been naught ie. continuously good

These are mutually exclusive & complementary events.

$$P(X_s = N) = 0.5 = 1 - P_{\overline{GG}}(0,t)$$

$$0.5 = 1 - P_{\overline{GG}}(0,t)$$

$$P_{\overline{GG}}(0,t) = 0.5$$

$$\exp[-(0.2t + 0.02t^2)] = 0.5$$

Taking ln on both sides →

$$-0.2t - 0.02t^2 = \ln 0.5$$

$$0.2t + 0.02t^2 - \ln 0.5 = 0$$

use the formula method

$$ax^2 + bx + c = 0$$

$$\Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$t = \frac{-0.2 + \sqrt{0.2^2 - 4 \times 0.02 \times \ln 0.5}}{2 \times 0.02}$$

$= 2.72$ hrs or 2hrs and 43 mins

KFD:

$$\frac{d P_{GG}(0,t)}{dt} = \sum_{k \in S.S.} P_{GK}(0,t) \cdot \mu_{KG}(t)$$

$$\boxed{\frac{d P_{ij}(s,t)}{dt} = \sum_{k \in S.S.} P_{ik}(s,t) \cdot \mu_{kj}(t)}$$

$$= P_{GN}(0,t) \cdot \mu_{NG}(t) + P_{GG}(0,t) \cdot \mu_{GG}(t)$$

$$= \underline{P_{GN}(0,t)} \left[0.4 - 0.0\cancel{4}t\right] + P_{GG}(0,t) \cdot \left[-0.2 - 0.04t\right]$$

Now, $P_{GN}(0,t) + P_{GG}(0,t) = 1 \quad \therefore P_{GN}(0,t) = \underline{1 - P_{GG}(0,t)}$

$$\mu_{ii} = -\sum_{i \neq j} \mu_{ij}, \quad \mu_{GG} = -\mu_{GN} = \underline{-0.2 - 0.04t}$$

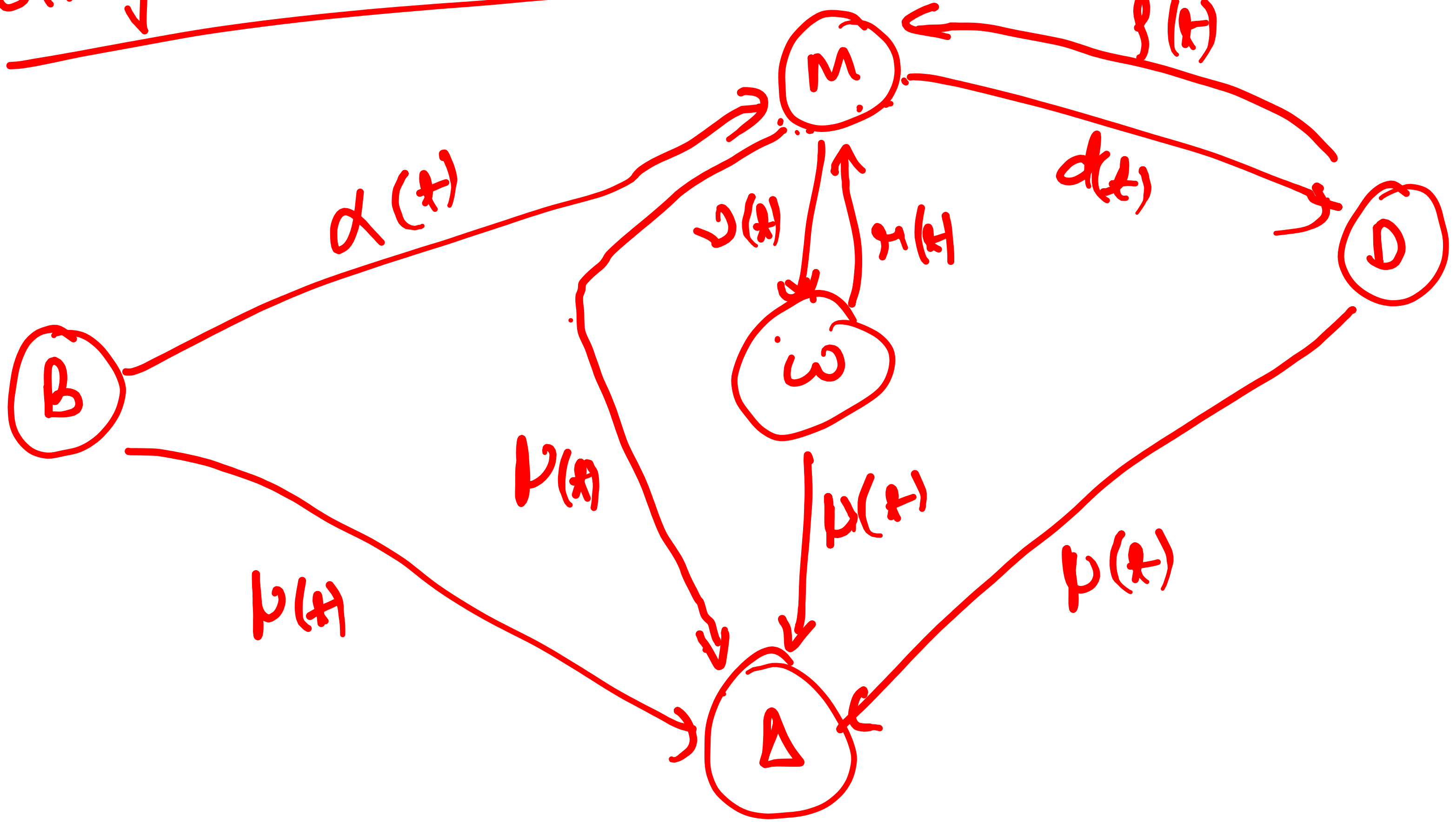
$$\therefore \frac{dP_{GG}(0,t)}{dt} = [1 - P_{GG}(0,t)]\cdot[0.4 - 0.04t] + P_{GG}(0,t)[-0.2 - 0.04t]$$

$$= 0.4 - 0.04t - 0.4P_G(t) + 0.04t \cdot P_G(t) - 0.2P_G(t) - 0.04t \cdot P_G(t)$$

$$\boxed{\frac{dP_G(t)}{dt} = 0.4 - 0.04t - 0.6P_G(t)}$$

Applications of Time inhomogen. MJ process

① Marriage model



B - Bachelor

M - Married

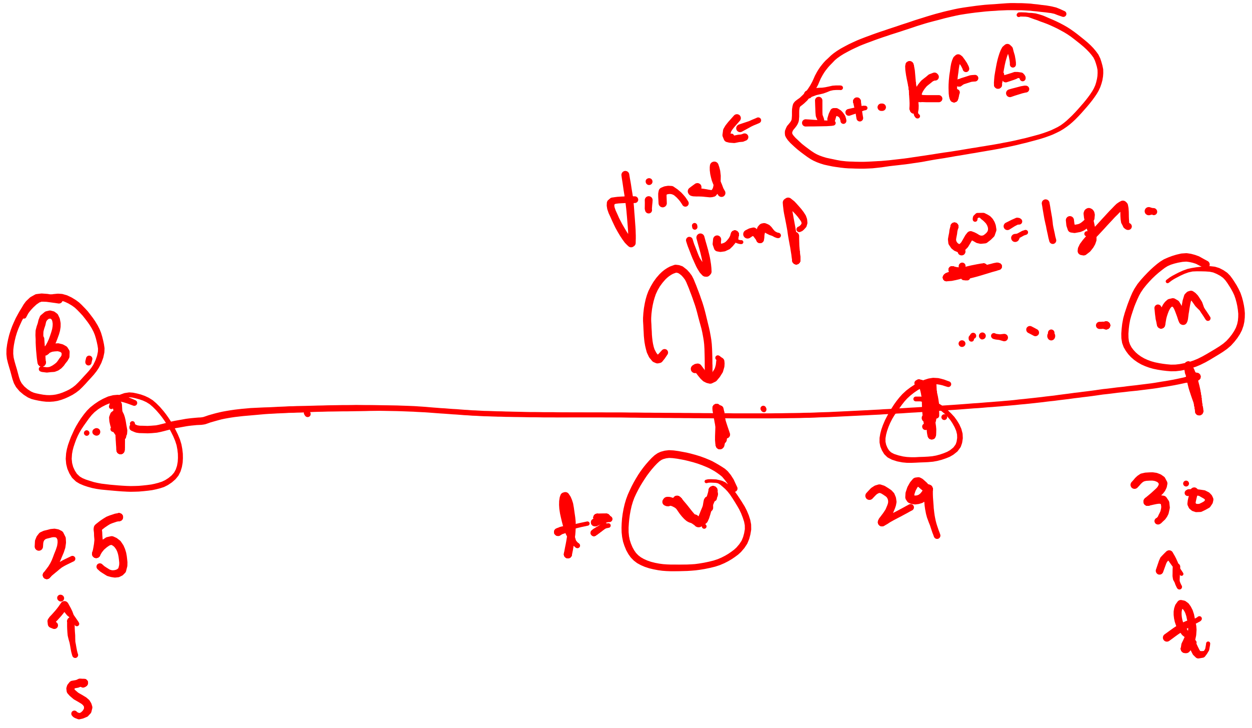
W - Widow

D - Divorced

Δ - Dead

$$P\left(X_t = M, \underline{C_t > w} \mid X_s = B\right) = \sum_{k \neq M} \int_{w}^{t-s} P_{Bk}(s, t-v) \cdot \mu_{kM}(t-v) \cdot \underline{P_{\overline{MM}}(t-v, t) \cdot dv}$$

Ex.



$$1 \le v \le 5$$

$$w \le v \le t-s$$

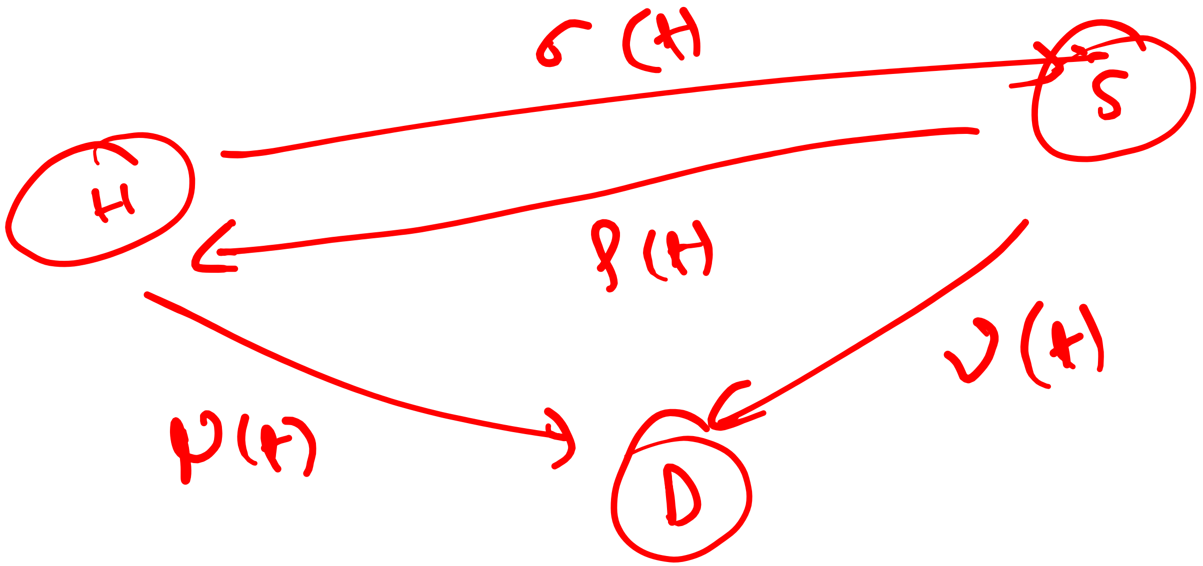
$$P\left(X_t = M,\ C_t > \omega \,/\, X_s = B\right)$$

$$= \int_{\omega}^{t-s} \Big[P_{BB}(s, t-v)\cdot \mu_{BM}(t-v) + P_{BD}(s, t-v)\cdot \mu_{DM}(t-v) + P_{BW}(s, t-v)\cdot \mu_{WM}(t-v) \Big] \cdot P_{\overline{MM}}(t-v, t)\, dv$$

$$\boxed{\begin{aligned} &\lambda_M(t) \\ &= \mu(t) + \nu(t) \\ &\quad + d(t) \end{aligned}}$$

$$= \int_{\omega}^{t-s} \Big[P_{BB}(s, t-v)\cdot \alpha(t-v) + P_{BD}(s, t-v)\cdot \beta(t-v) + P_{BW}(s, t-v)\cdot \eta(t-v) \Big] \cdot \exp\left[-\int_{t-v}^{t} \left(\mu(s) + \nu(s) + d(s)\right) ds \right] dv$$

Sickness and Death model with duration

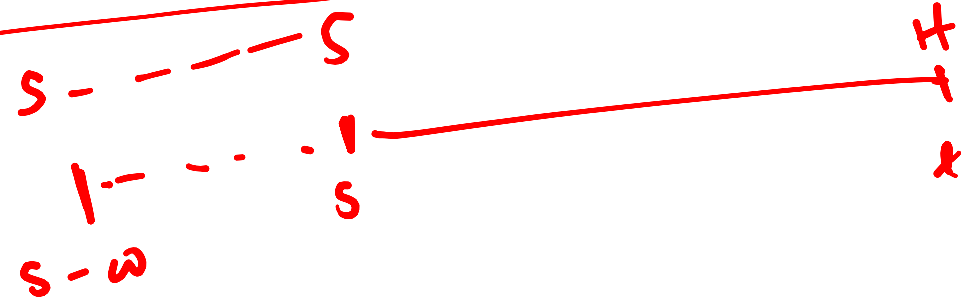


$$P[X_t = H, C_t > w \mid X_s = S] \leftarrow$$ we can do this same as before

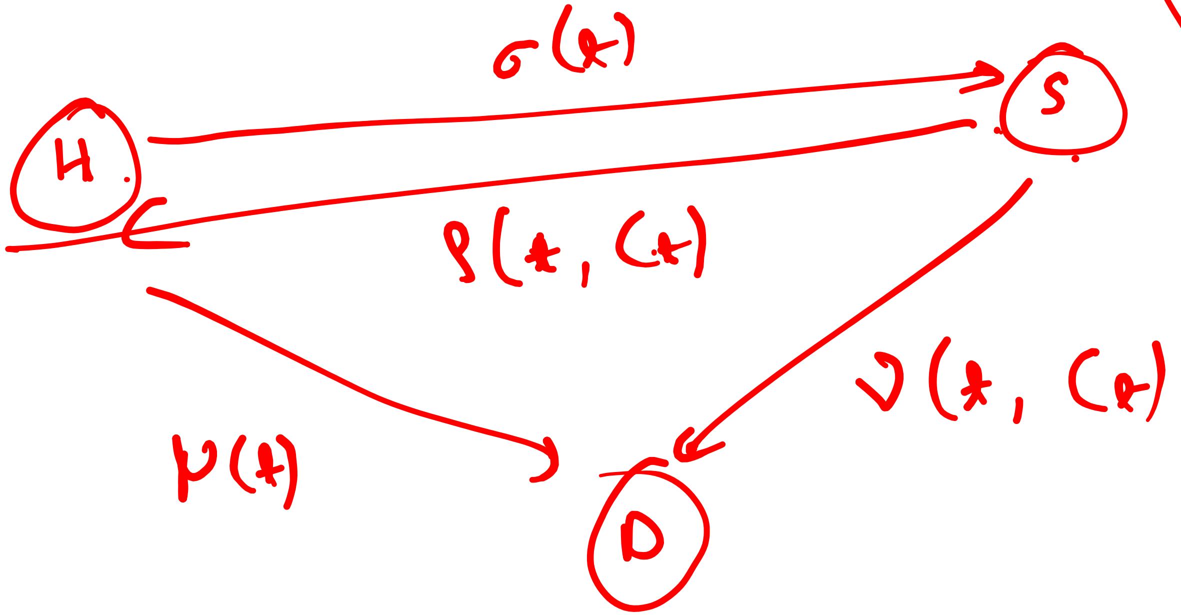
$$P[X_t = H / X_s = S, \underline{C_s > w}] = P[X_t = H / X_s = S]$$

→ because of Markov property

Although, current holding time in sick state does not impact future recovery prob. as per Markov property, it is logical to include them. We update our Markov process to do this.



Updated model:



Transition intensities out of sick state will depend upon both starting time (age and current holding time.

$\rho(t, C_t) \to$ Decreasing f^n of C_t

$\nu(t, C_t) \to$ Increasing f^n of C_t.