INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

**Subject:** Statistical Techniques and Risk Modelling -IV

**Chapter:** Unit 1&2

**Category:** Assignment

1.

**iv)** Let Ti be the time until default of bond i where i = 1,2,3 . We want to calculate the joint probability:

P(T1 <=1,T2 <=1,T3 <=1)= C[u1,u2,u3]

Where $u_i$ =P[$T_i$ <=1]=0.1 for i=1,2,3

[1]

Using the Gumbel copula with parameter α= 2 , we have:

P(T1 <=1,T2 <=1,T3 <=1)=c[u1,u2,3]

$=\text{Exp}\{-((-\ln u1)^2+(-\ln u2)^2+(-\ln u3)^2)^{1/2}\}$ [2]

=0.0185 or 1.85% [1]

**[4]**

**v)**

- The Gumbel copula exhibits (non-zero) upper-tail dependence, the degree of which can be varied by adjusting the single parameter. However, it exhibits no lower tail dependence.
- Hence, the Gumbel copula is appropriate if we believe that the three investments are likely to behave similarly as the term approaches five years but not at early durations.
- This is unlikely to be the case though. If one bond defaults early on, then it may be indicative of problems in the industry sector or the economy and so the other investments may also be likely to default early on.
- If we believe the performance of investments issued by companies within the industry are much more closely associated (eg subject to the same systemic and operational risk factors), then a copula that exhibits both lower and upper tail dependence, such as the Student's t copula, may be more appropriate.

[1 each = 4]

2.

**i)** The three parameters of GEV are

a) A location parameter $\alpha$

b) A scale parameter $\beta$, this is always greater than zero

c) A shape parameter $\gamma$

[2]

**ii)** The first two parameters rescale the distribution and determine shift and stretch of the distribution. The third parameter determines overall shape of the distribution and its sign. [2]

**iii)** These are analogous to following in the standard distribution

1. Mean
2. Standard deviation
3. Skewness

[2]



3.

**i)** The four ways of measuring the tail weight of a distribution:

- The existence of moments
- Limiting density ratios
- The hazard rate
- The mean residual life [1]

**ii)** $$\lim_{x\to\infty}\frac{f_{\alpha=0.2}(x)}{f_{\alpha=0.3}(x)} = \lim_{x\to\infty}\left\{\frac{0.2\lambda^{0.2}}{(\lambda+x)^{1.2}}\Big/\frac{0.3\lambda^{0.3}}{(\lambda+x)^{1.3}}\right\}$$

$$= \frac{2}{3} * \frac{1}{\lambda^{0.1}}\lim_{x\to\infty}(\lambda+x)^{0.1}$$

$$= \infty$$

Then the distribution with $\alpha = 0.2$ has a much thicker tail. [2]

**[3 Marks]**

4.

Given that ${}_{30}P_{50}$ = 0.6 for Male and ${}_{30}P_{50}$ = 0.65 for Female.

⇨ The probability of death for the same period is
${}_{30}q_{50}$ = 0.4 for Male and ${}_{30}q_{50}$ = 0.35 for Female

Let X be the random variable representing the future lifetime of Male and Y be the random variable representing the future lifetime of Female. Then

$P(X \le 30) = 0.4$ and $P(Y \le 30) = 0.35$.

What we require is $P(X \le 30, Y \le 30)$.

**i)** The Gumbel Copula with α = 3.5

u = 0.4 and v = 0.35

u = 0.4 and v = 0.35

$$C[u, v] = \exp\left\{-\left((-\ln u)^{\wedge}\alpha + (-\ln v)^{\wedge}\alpha\right)^{\wedge}(1/\alpha)\right\}$$

Applying the values of α, u &v , C [u, v] = 0.2996 [2]



**ii)** The Clayton Copula with α = 3.5

$C[u, v] = (u^{-\alpha} + v^{-\alpha} - 1)^{-1/\alpha}$

Applying the values of α, u &v , C [u, v] = 0.30595 [1]

**iii)** The Frank copula with α = 3.5

$$C[u, v] = (-1/\alpha) \ln \left\{ \frac{1 + (e^{-\alpha u} - 1)(e^{-\alpha v} - 1)}{(e^{-\alpha} - 1)} \right\}$$

Applying the values of α, u &v , C [u, v] = 0.2273 [1]

**iv)** Clayton Copulas gives the highest probability of claims because it exhibits lower tail dependence. This means that if one life does not survive for long then there is high probability that the other life will also not survive for long.

If the deaths are independent then the probability of paying the benefit is 0.14.

However, two lives covered are related, so we would expect the probability of paying the benefit to be higher than under the assumption of independence. Hence Clayton copula is more appropriate.

[2]



5.

**i)** Mean Residual Life of two year old phone

$$= \frac{\int_x^\infty \{1 - F(y)\} dy}{1 - F(x)}$$ [1]

$$= \int_x^\infty e^{-\lambda y} dy \,/\, e^{-\lambda x}$$ [1]

$= 1/\lambda$

= 8 years. [1]

**[3]**

**ii)**

The expected life time of a new phone = 1/λ = 8 years. [0.5]

And the expected life time of a two year old phone is also 8 years. [0.5]

This is due to the memory loss property of exponential distribution. [0.5]

Hence the exponential distribution is not appropriate for measuring Mean Residual Life of a product.

[0.5]

**[2]**

**[5 Marks]**

6.

**i)** When t=0, $\psi(0) = \lim_{t \to 0}(-\ln t)^{\alpha} = \infty$

Therefore, it is a strict generator function.

The inverse function is found by rearranging the equation:

x= $\psi$(t) = (-ln t)$^{\alpha}$

-ln t= x^( 1/α)

t= exp(-x ^(1/ α))

ii) C[u,v] = $\psi^{-1}[\psi(u) + \psi(v)]$

$$= \psi^{-1}[(-\ln u)^{\alpha} + (-\ln v)^{\alpha}]$$

$$= \exp\{- ( (-\ln u)^{\alpha} + (-\ln v)^{\alpha} ) \wedge (\tfrac{1}{\alpha})\} \text{ for } \alpha \geq 1$$

The coefficient of lower tail dependence is given by:

$$\lambda_L = \lim_{u \to 0+} \frac{c[u,u]}{u}$$

$$= \lim_{u \to 0+} [\exp\{-( (-\ln u)^{\alpha} + (-\ln u)^{\alpha} ) \wedge (\tfrac{1}{\alpha})\}/u]$$

$$= \lim_{u \to 0+} [\exp\{-(2 \wedge (\tfrac{1}{\alpha}) \ (-\text{In } u) \}/u]$$

$$= \lim_{u \to 0+} [\exp\{ (2 \wedge (\tfrac{1}{\alpha}) \ (\text{In } u) \}/u$$

$$= \lim_{u \to 0+} (u^{2^{\frac{1}{\alpha}}}) \ /u$$

$$= \lim_{u \to 0+} u^{2^{\frac{1}{\alpha}}-1}$$

$$= 0$$