INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical Techniques and Risk Modelling -4

Chapter: Unit 3&4

Category: Assignment

Q1.

Y_1 follows the below process:

$$\gamma_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2\left(Y_t - \beta_1 e_t^2 Y_{t-1}\right)\mu - \left(1 - \beta_1 e_t^2\right)\mu^2 + \beta_0 e_t^2$$

Where e_t are independent (from $Y_{t'}, Y_{t-1,}$...) standard normal random variable.

i) Calculate $Cov\left(Y_{t'}, Y_{t-s}\right)$

ii) Find out whether Y_t and Y_{t-s} are dependent.

Now X_t follows below process:

$$X_t = 0.5Y_t + 0.3t + 0.1$$

iii) Find out whether first order difference of X_t is stationary.

Q2.

i) Consider the time series model:

$$Y_t = Y_{t-1} + 0.5Y_{t-2} = 0.5Y_{t-3} + Z_t + 0.3Z_{t-1}$$

where $\{Z_t\}$ is a sequence of uncorrelated random variables each having the normal distribution with mean zero and variance σ^2.

Show that the above model is a special case of the ARIMA (p, d, q) model and determine p, d and q.

(ii) Let $X = (1 - B)^d Y$. Determine whether $\left\{X_t\right\}$ is a stationary time series.

(iii) Calculate the autocorrelation function of $\left\{X_t\right\}$

Q3.

(i) The following time series model is used for the monthly inflation rate $\{Y_t\}$ in India, tased on wholesale price of select commodities.

$$Y_t = 0.4Y_{t-1} + 0.2Y_{t-2} + Z_t + 0.025Z_{t-1} + 0.016.$$

Here, $\{Z_t\}$ is a sequence of uncorrelated random variables having a common variance and zero mean.
(a) Obtain the values of p, d and q for this ARIMA(p, d, q) model.
(b) Assuming infinite history, calculate the expected value of the inflation rate according to this model.
(c) Determine whether $\{Y_t\}$ is a stationary process.

(ii) It was thought that the monthly inflation rate $\{X_t\}$ in India, based on retail price of select commodities would follow the model

$$X_t = 0.4X_{t-1} + 0.2X_{t-1} + Z_t + 0.025$$

where $\{Z_t\}$ is a sequence of uncorrelated random variables having a common variance and zero mean.
(a) Determine the autocorrelation function of $\{X_t\}$.
(b) Describe briefly two diagnostic checks that *can* be performed to detcrmine whether there is any inadequacy in the fitting of such ARMA models to time series data.

Q4.

Consider the AR(2) process

$$Y_t = -\alpha Y_{t-1} + \alpha^2 Y_{t-2} + Z_t$$

(i) Determine the range of values of α for which the above process can be stationary.
(ii) Derive the auto-covariance γ_1 & γ_2 of Y in terms of α and σ

Q5.

The following data summary is obtained from 200 consecutive points of a realization of an ARIMA $(1, 1, 0)$ process with constant mean.

$$\sum_{i=2}^{300} \left(x_i - x_{i-1}\right)^2 = 93649, \sum_{i=2}^{199} \left(x_i - x_{i-1}\right)\left(x_{i+1} - x_i\right) = 587.83$$

(i) Obtain the usual estimates of the AR coefficient and the innovation variance from the given data summary.

(ii) Given the data points $x_{198} =- 2.11, x_{199} = 0.82$ and $x_{200} = 1.93$, use the Box Jenkins methodology and the estimated parameters obtained in part (i) to forecast x_{201}.

Q6.

i) Give four different purposes, for which one carries out time series analysis.

ii) State whether each of the following processes is Markov or not.

a. An MA (1) process:

b. $An\ AR(1)$ procens;

c. An $AR(2)$ process:

d. A random walk.

iii) Suppose that $Y(t)$ is a stationary time series and $X(t) = a + bt + Y(t)$, where a and b are fixed constants, Indicate, with reason, whether

a. $X(t)$ in stationary.

b. $X(t)$ is $I(d)$ for any value of d.

iv) Let $X_1(l)$ and $X_2(l)$ be time series, which are $I(1)$, and satisfy the relations

$$X_1(t) = aX_1(t - 1) + bX_2(t - 1) + e_1(t).$$

$$X_2(t) = bX_1(t - 1) + aX_2(t - 1) + e_2(t).$$

where a and b the non-zero constants, and $e_1(t)$ and $e_2(t)$ are white noise processes.

Can the two time series be said to be cointegrated? Explain.

Q7.

i) Define, giving relevant equations, the three main linear models used for modeling stationary time series.
(6)
ii) Classify each of the following processes as $ARIMA(p, d, q)$, if possible:
a) $X_4 = 0.6\varepsilon_{t-1} + \varepsilon_t$
b) $X_4 = 1.4X_{t-2} + \varepsilon_t + 0.5\varepsilon_{1-3}$
c) $X_t = 1.4X_{t-1} = 0.4X_{t-2} + \varepsilon_t + \varepsilon_{t-1}$
In each case ε_t denotes white noise with mean 0 and variance σ^2.

iii) A moving average time series is defined by the relationship

$$X_t = 3.1 + \varepsilon_t + 0.25\varepsilon_{t-1} + 0.5\varepsilon_{t-2} + 0.25\varepsilon_{t-3}$$

where $\varepsilon_t \sim N(0, \sigma^2)$ denotes white noise.
Calculate the autocorrelation function ρ_k, at all lags k.

Q8.

i) $X_t = A_t + B_t$;

$$A_t = 0.5X_{t-1} + e_t^{(A)}$$

$$B_t = 0.7\left(A_{t-1} - A_{t-2}\right) + e_t^{(B)};$$

$$Y_t = \left(\frac{A_t}{B_t}\right)$$

where $e_t^{(A)}$ and $e_t^{(B)}$ are zero mean white noise process
Determine whether Y_t is a stationary process.

ii) A time series model is given by the following equation:

$$X_t = (\alpha + 1)X_{t-1} - \left(\alpha + 0.25\alpha^2\right)X_{t-2}, + 0.25\alpha^2 X_{t-3} + e_t$$

where e_t is a white noise process with variasce σ^2

a) Check whether the above can be expressed as ARIMA (p,d,v) process, specifying the range of α
b) if the autocovariance function $\left(\gamma_K\right)$ of the above ARIMA process follows the below equation:

$$\lambda_1 + k\lambda_2 - (0.5\alpha)^{-K}\gamma_K,$$

then find λ_1 and λ_2 in terms of α and σ^2

c) If $x_1 x_2 \ldots x_{50}$ are the observed values of X_t then forecast the next two observation x_{51} and x_{52}, assume $\alpha = 0.04$.

Q9.

Consider the ARCH(1) process:

$$X_t = \mu + e^t\sqrt{\alpha + \beta\left(X_{t-1} - \mu\right)^2}$$

Where $e\ t$ are independent normal random variables with variance 1 and mean 0. Show that, for s = 1, 2, ... , t-1, X_t & X_{t-1} are:

i) Uncorrelated

ii) Not independent

Q10.

i) The autocorrelation function of a time series shows coefficients significantly different from zero at lags 1 through 4. The partial autocorrelation function shows one spike and monotonically increases to zero as lags length increases. If the time series is model by ARMA (p, q) then state values of p and q.

ii) State the definitions of four non-linear time series models.

iii) If X_t follows $MA(1)$ and $Yt = 0.6 + 0.3t + X_t$ then prove that the standard deviation of first difference of Y_t will be higher than that of X_t.

Q11.

Consider the time series model defined by: $Y_t = \alpha_1 Y_{t-1} + \alpha_2 Y_{t-2} + \alpha_3 Y_{t-3} + \epsilon_t$

i) Derive the formula for autocorrelation coefficient with lag 1 and lag 2 in terms of $\alpha 1$, $\alpha 2$ and $\alpha 3$.

ii) By considering $\alpha 1$ = $\alpha 2$ = $\alpha 3$= 0.3, compute the value of autocorrelation coefficient with lag 1 and lag 2.

iii) From (ii) or otherwise, compute the PACF with lag 1 and lag 2.

iv) Comment on the ACF and PACF values computed in (ii) and (iii) above.

Q12.

i) Explain with the help of examples how the lack of stationarity in a time series may be caused?

ii) A student is using the Box Jenkins approach to model $X(t)$, an observed time series of a ARIMA($p,\ d, q$) process.

a) Write down the general equation for $X(t)$, if it is a ARIMA(p,d,q) process.

b) What are the main steps involved in the Box-Jenkins methodology.

c) The following table shows information regarding the dth order difference of the observed time series

Sample Autocorrelation coefficient	$d = 1$	$d = 2$	$d = 3$
$r1$	-0.32	-0.60	-0.80
$r2$	-0.02	-0.47	0.54
$r3$	0.12	0.15	-0.12
$r4$	0.20	0.25	0.20
Sample variance	5.0	50	120

Explain with reasons which value of d you consider most appropriate if this series is to be modelled using an ARIMA(p,d,q) model.

d) Classify each of the following processes as ARIMA (p, d, q) process, where e_t denotes the white noise with mean $= 0$ and variance $= \sigma^2$

i. $X_t = 0.8e_{e-1} + e_t$

ii. $X_1 = 2X_{t-1} + e_t + 0.5e_{t-4}$

iii. $X_t = 1.5X_{t-1} + 0.2X_{t-2} + e_{t-1} + e_{t-2}$

Q13.

(i) Define the term "(weakly) stationary".

(ii) Show that the process $\{X_n\}$ satisfying the equation

$X_n = Z_n + \beta Z_{n-1}$,

where $\{Z_n\}$ is a white noise process with mean μ and variance σ^2, is weakly stationary?

(iii) Is the process $\{X_n\}$ satisfying the equation

$X_n = 1.5X_{n-1} - 0.5X_{n-2} + Z_n$,

where $\{Z_n\}$ is a white noise process with mean 0 and variance σ^2, weakly stationary?

(iv) Determine the autocorrelation at lag 1 of a suitably differenced version of $\{X_n\}$ in (iii) above.