INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: Statistical Techniques and Risk Modelling- 4

Chapter: Unit 3 & 4

Category: Assignment

Q1.

Solution 1:

i)

$$Y_t^2 - \beta_1 e_t^2 Y_{t-1}^2 = 2(Y_t - \beta_1 e_t^2 Y_{t-1})\mu - (1-\beta_1 e_t^2)\mu^2 + \beta_0 e_t^2$$

Or, $Y_t^2 - 2Y_t\mu + \mu^2 = e_t^2(\beta_0 + \beta_1 Y_{t-1}^2 - 2\beta_1 Y_{t-1}\mu + \beta_1\mu^2)$

Or, $(Y_t - \mu)^2 = e_t^2(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)$

Or, $Y_t = \mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}$

Now,

$$E(Y_t) = E(\mu) + E(e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$$

e_t and Y_{t-1} are independent.

$$E(Y_t) = \mu + E(e_t)E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$$

Or, $E(Y_t) = \mu + 0 \times E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})$

Hence, $E(Y_t) = \mu$

Now,

$$Cov(Y_t, Y_{t-s}) = E(Y_t Y_{t-s}) - E(Y_t)E(Y_{t-s})$$

Or, $Cov(Y_t, Y_{t-s}) = E((\mu + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5})(\mu + e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5})) - \mu^2$

Or, $Cov(Y_t, Y_{t-s}) = E(\mu^2 + \mu e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5} + \mu e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5} + e_t(\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5} e_{t-s}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

Or, $Cov(Y_t, Y_{t-s}) = \mu^2 + \mu E(e_t)E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}) + \mu E(e_{t-s})E((\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) + E(e_t)E(e_{t-s})E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

Or, $Cov(Y_t, Y_{t-s}) = \mu^2 + 0 + 0 + 0 \times E((\beta_0 + \beta_1(Y_{t-1} - \mu)^2)^{0.5}(\beta_0 + \beta_1(Y_{t-s-1} - \mu)^2)^{0.5}) - \mu^2$

Or, $Cov(Y_t, Y_{t-s}) = 0$ [6]

ii)

Now,

$$Var(Y_t \mid Y_{t-1}) = Var(e_t)Var(\beta_0 + \beta_1(Y_{t-1} - \mu)^2) = \beta_0 + \beta_1(Y_{t-1} - \mu)^2$$

From the above equation we can see that variance of Y_t depends on Y_{t-1}. Similarly recursively we can see that variance of Yt will depend on Y_{t-s}. Hence Y_t and Y_{t-s} are dependent. [2]

iii)

The first difference of Xt can be written as given below:

$$\Delta X_t = X_t - X_{t-1}$$

Now,

$$E(\Delta X_t) = E(X_t) - E(X_{t-1})$$

Or, $E(\Delta X_t) = E(0.5Y_t + 0.3t + 0.1) - E(0.5Y_{t-1} + 0.3(t-1) + 0.1)$

Or, $E(\Delta X_t) = 0.5\mu + 0.3t + 0.1 - 0.5\mu - 0.3(t-1) - 0.1 = 0.3$

The mean is independent of t and hence constant.

Now,

$$Cov(\Delta X_t, \Delta X_{t-s}) = Cov(X_t - X_{t-1}, X_{t-s} - X_{t-s-1})$$

Or, $Cov(\Delta X_t, \Delta X_{t-s}) = Cov(0.3 + Y_t - Y_{t-1}, 0.3 + Y_{t-s} - Y_{t-s-1})$

Or, $Cov(\Delta X_t, \Delta X_{t-s}) = Cov(Y_t - Y_{t-1}, Y_{t-s} - Y_{t-s-1})$

Or, $Cov(\Delta X_t, \Delta X_{t-s}) = Cov(Y_t, Y_{t-s}) - Cov(Y_t, Y_{t-s-1}) - Cov(Y_{t-1}, Y_{t-s}) + Cov(Y_{t-1}, Y_{t-s-1})$

Or, $Cov(\Delta X_t, \Delta X_{t-s}) = 0 - 0 - 0 + 0 = 0$

The auto covariance function is constant hence the first difference of X_t is stationary.

[5]

Q2.

(i) The characteristic equation is

$$1 - z - .5z^2 + .5z^3 = 0.$$

The cubic polynomial of the left hand side factorizes as $(1-z)(1-.5z^2)$. There is exactly one root on the unit circle. Therefore, $d = 1$. [1]

Rewriting the model in terms of $X = (1 - B)Y$, we have

$$X_t - .5X_{t-2} = Z_t + .3Z_{t-1},$$

which is ARMA(2,1). Thus, the model for Y_t is ARIMA(2,1,1).[1]

(ii) The characteristic polynomial of X is $(1 - .5z^2)$, whose roots are $\pm\sqrt{2}$. As the roots are outside the unit circle, the process $\{X_t\}$ is stationary. [2]

(iii) The model equation is $X_t = .5X_{t-2} + Z_t + .3Z_{t-1}$. By taking covariances of both sides of this equation with Z_t, Z_{t-1} and Z_{t-2}, we have

$$\begin{aligned}
cov(X_t, Z_t) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_t) \\
&= 0 + \sigma^2 + 0 = \sigma^2, \\
cov(X_t, Z_{t-1}) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_{t-1}) \\
&= 0 + 0 + .3\sigma^2 = .3\sigma^2, \\
cov(X_t, Z_{t-2}) &= cov(.5X_{t-2} + Z_t + .3Z_{t-1}, Z_{t-2}) \\
&= .5\sigma^2 + 0 + 0 = .5\sigma^2.
\end{aligned}$$

[2]

By taking covariances of both sides of the model equation with X_t, X_{t-1}, X_{t-2} and X_{t-k} (for $k > 2$), we have

$$\begin{aligned}
\gamma(0) &= cov(X_t, X_t) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_t) \\
&= .5\gamma(2) + \sigma^2 + .09\sigma^2 = .5\gamma(2) + 1.09\sigma^2, && (1) \\
\gamma(1) &= cov(X_t, X_{t-1}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-1}) \\
&= .5\gamma(1) + 0 + .3\sigma^2 = .5\gamma(1) + .3\sigma^2, && (2) \\
\gamma(2) &= cov(X_t, X_{t-2}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-2}) \\
&= .5\gamma(0) + 0 + 0 = .5\gamma(0), && (3) \\
\gamma(k) &= cov(X_t, X_{t-k}) = cov(.5X_{t-2} + Z_t + .3Z_{t-1}, X_{t-k}) \\
&= .5\gamma(k-2) + 0 + 0 = .5\gamma(k-2), \quad k > 2. && (4)
\end{aligned}$$

[2]

By substituting for $\gamma(2)$ from (3) into (1), we have $\gamma(0) = .25\gamma(0) + 1.09\sigma^2$, i.e., $\gamma(0) = 109\sigma^2/75$. Equation (2) implies $\gamma(1) = 3\sigma^2/5$. Thus, $\rho(1) = \gamma(1)/\gamma(0) = 45/109$. Equations (3) and (4) together imply $\rho(k) = .5\rho(k-2)$ for $k \geq 2$. It follows that

$$\rho(k) = \begin{cases} (.5)^{|k|/2} & \text{if } |k| \text{ is even,} \\ (45/109)(.5)^{(|k|-1)/2} & \text{if } |k| \text{ is odd.} \end{cases}$$ [2]

Q3.

(i) (a) The process can be written as $(1-0.4B-0.2B^2)Y_t = (1+0.025B)Z_t + 0.016$.

The characteristic equation is $1-0.4z-0.2z^2 = 0$.

There is no root having magnitude 1. Therefore, $d = 0$.

Hence, the process is ARIMA (2,0,1).

(b) $(1-0.4-0.2)E(Y_t) = 0.016$. Therefore, $E(Y_t) = 0.016/0.4 = 0.04$ or 4%.

(c) The two roots of the characteristics equation are $-1 \pm \sqrt{6}$, i.e., 1.4495 and –3.4495, both of which have magnitude larger than 1. Hence, the process $\{Y_t\}$ is stationary.

(ii) (a) This AR process has the same characteristic equations and the same roots as in part (iii). The k^{th} order auto-covariance is of the form (given in Core Reading, chapter CT6-12, section 3.4)

$$\gamma_k = A_1(-1+\sqrt{6})^{-k} + A_2(-1-\sqrt{6})^{-k},$$

where A_1 and A_2 are constants. Therefore, the k^{th} order auto-correlation is of the form

$$\rho_k = \frac{\gamma_k}{\gamma_0} = \frac{A_1(-1+\sqrt{6})^{-k} + A_2(-1-\sqrt{6})^{-k}}{A_1 + A_2} = \alpha(-1+\sqrt{6})^{-k} + (1-\alpha)(-1-\sqrt{6})^{-k}, \text{ w}$$

here α is a constant. We can determine α by calculating ρ_1 directly from the Yule-Walker equation

$\gamma_1 = 0.4\gamma_0 + 0.2\gamma_1$,

which implies that $\rho_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{0.4}{1-0.2} = 0.5$. Equating this value with the general expression for $k = 1$, we have

$0.5 = \alpha(-1+\sqrt{6})^{-1} + (1-\alpha)(-1-\sqrt{6})^{-1}$, and therefore,

$$\alpha = \frac{0.5-(-1-\sqrt{6})^{-1}}{(-1+\sqrt{6})^{-1}-(-1-\sqrt{6})^{-1}} = \frac{2\sqrt{6}+3}{4\sqrt{6}} = 0.8062.$$

It follows that

$$\rho_k = \left(\frac{2\sqrt{6}+3}{4\sqrt{6}}\right)(-1+\sqrt{6})^{-k} + \left(\frac{2\sqrt{6}-3}{4\sqrt{6}}\right)(-1-\sqrt{6})^{-k}$$

$$= 0.8062(0.6899)^k + 0.1938(-0.2899)^k.$$

The ACF values for the first few lags are $\rho_1 = 0.5$, $\rho_2 = 0.4$, $\rho_3 = 0.26$, $\rho_4 = 0.184$.

(Partial credit for determining a few ACF values: 1 mark for correct computation of each value. Maximum partial credit with no general solution is 3.)

(b) Three diagnostic checks are as under (**any two** should fetch full credit).

- *Inspection of the graph of the time-plot of residuals*: Visual inspection might reveal a pattern, such as uneven fluctuations or clusters of only positive / only negative residuals, which indicate inadequate fit.

- *Inspection of the sample autocorrelation functions of the residuals*: Too many ACF or PACF values outside the range $\pm 2/\sqrt{N}$ (N being the sample size) may indicate poor fit or too few parameters.

- *Counting turning points*: The number of turning points (points where the value of the time series is smaller/larger than both neighboring values) for a sequence of independent random variables has average $2(N-2)/3$ and variance $(16N-29)/90$. If the residuals from a particular fit has too few or too many turning points (with reference to a normal distribution with the said mean and variance), then the fit is inadequate.

[12]

Q4.

(i) The model can be written in the form:

$(1 + \alpha B - \alpha^2 B^2) Y_t = Z_t$

So the characteristic equation is:

$1 + \alpha x - \alpha^2 x^2 = 0.$

Applying the quadratic formula, we see that this has roots:

$x = (1 \pm \sqrt{5}) / 2\alpha.$

Roots must lie outside the unit circle, so we require that $(\sqrt{5}-1)/2.|\alpha| > 1$ and $(\sqrt{5}+1)/2.|\alpha| > 1$, which means, $|\alpha| < (\sqrt{5} - 1) / 2.$

(ii) $Y_t = -\alpha\, Y_{t-1} + \alpha^2\, Y_{t-2} + Z_t$

The Yule-Walker equations are:

$\text{Cov}[Y_t, Y_t] = \gamma_0 = -\alpha.\gamma_1 + \alpha^2 \gamma_2 + \sigma^2.$ ----- Equation (1)

$\text{Cov}[Y_t, Y_{t-1}] = \gamma_1 = -\alpha.\gamma_0 + \alpha^2 \gamma_1.$ ----- Equation (2)

$\text{Cov}[Y_t, Y_{t-2}] = \gamma_2 = -\alpha\,\gamma_1 + \alpha^2 \gamma_0.$ ----- Equation (3)

From equation (2), $\gamma_1 = -\alpha.\gamma_0 / (1 - \alpha^2).$ ----- Equation (4)

Substituting the value of γ_1 in equation (3), we have

$\gamma_2 = -\alpha.\,[-\alpha.\gamma_0 / (1-\alpha^2)] + \alpha^2.\,\gamma_0 = (2\alpha^2 - \alpha^4).\,\gamma_0 / (1 - \alpha^2).$ ----- Equation (5)

Substituting the values of γ_1 and γ_2 in equation (1), we have

$\gamma_0 = -\alpha(-\alpha.\gamma_0 / (1 - \alpha^2)) + \alpha^2.(2\alpha^2 - \alpha^4).\,\gamma_0 / (1 - \alpha^2) + \sigma^2.$

Thus,

$\gamma_0 = \sigma^2.\,(1 - \alpha^2) / (1 - 2\alpha^2 - 2\alpha^4 + \alpha^6).$

Substituting the value of γ_0 in equations (4) and (5), we have

$\gamma_1 = -\sigma^2.\,\alpha / (1 - 2\alpha^2 - 2\alpha^4 + \alpha^6).$

$\gamma_2 = \sigma^2.\,(2\alpha^2 - \alpha^4) / (1 - 2\alpha^2 - 2\alpha^4 + \alpha^6).$

Q5.

(i) The model is

$(X_n - X_{n-1}) = \alpha(X_{n-1} - X_{n-2}) + \varepsilon_n$, ε_n's are uncorrelated, with mean 0 and variance σ^2.

Since the mean of the process is known to be 0, the usual estimator of the parameter α is

$$\hat{\alpha} = \hat{\rho}_1 = \frac{\sum_{i=3}^{200}(X_i - X_{i-1})(X_{i-1} - X_{i-2})}{\sum_{i=2}^{200}(X_i - X_{i-1})^2} = \frac{587.83}{936.49} = 0.6277.$$

The estimator of autocovariance at lag 0 is

$$\hat{\gamma}_0 = \frac{1}{200}\sum_{i=2}^{200}(X_i - X_{i-1})^2 = \frac{936.49}{200} = 4.6825.$$

Using the relation $\gamma_0 = \dfrac{\sigma^2}{1-\rho_1^2} = \dfrac{\sigma^2}{1-\alpha^2}$, we estimate

$$\hat{\sigma}^2 = (1-\hat{\alpha}^2)\hat{\gamma}_0 = (1-0.6277^2)\times 4.6825 = 2.8376.$$

(ii) The forecast of x_{201} is obtained from

$$(\hat{x}_{200}(1) - x_{200}) = \hat{\alpha}(x_{200} - x_{199}) = 0.6277\times(1.93-0.82) = 0.6967.$$

Thus, $\hat{x}_{200}(1) = x_{200} + 0.6967 = 1.93 + 0.6967 = 2.6267$.

Q6.

(i) The purposes of a practical time series analysis may be summarized as:

- Description of the data;
- Construction of a model which fits the data;
- Forecasting future values of the process;
- Deciding whether the process is out of control, requiring action;
- For vector time series, investigating connections between two or more observed processes with the aim of using values of some of the processes to predict those of the others.

(ii) AR(1) and random walk are Markov.

MA(1) and AR(2) are not Markov.

(iii) **a.** $E[X(t)] = a + bt + E[Y(t)]$.

Since $Y(t)$ is stationary, $E[Y(t)]$ is equal to some constant c that does not depend on t.

Therefore, $E[X(t)] = a + bt + c$, which depends on t.

It follows that $X(t)$ cannot be stationary.

b. $E[\nabla X(t)] = E[X(t)] - E[X(t-1)] = a + bt + c - [a + b(t-1) + c] = b$.

Thus, the mean does not depend on t.

$$\begin{aligned}
Cov[\nabla X(t), \nabla X(s)] &= Cov[\{X(t) - X(t-1)\}, \{X(s) - X(s-1)\}] \\
&= Cov[\{b + Y(t) - Y(t-1)\}, \{b + Y(s) - Y(s-1)\}] \\
&= Cov[\{Y(t) - Y(t-1)\}, \{Y(s) - Y(s-1)\}] \\
&= Cov[Y(t), Y(s)] - Cov[Y(t), Y(s-1)] - Cov[Y(t-1), Y(s)] \\
&\quad + Cov[Y(t-1), Y(s-1)] \\
&= C_Y(t-s) - C_Y(t-s+1) - C_Y(t-s-1) + C_Y(t-s),
\end{aligned}$$

where C_Y is the covariance function of the stationary process $Y(t)$. The last expression depends on t and s only through the difference $t - s$. Hence, the process $\nabla X(t)$ is stationary.

As $\nabla X(t)$ is stationary but $X(t)$ is not, the process $X(t)$ is $I(1)$.

(iv) Compute the difference between the processes:

$$\begin{aligned}
X_1(t) - X_2(t) &= a[X_1(t-1) - X_2(t-1)] + b[X_2(t-1) - X_1(t-1)] \\
&\quad + [e_1(t) - e_2(t)] . \\
&= (a-b)[X_1(t-1) - X_2(t-1)] + [e_1(t) - e_2(t)] .
\end{aligned}$$

Let $Y(t) = X_1(t) - X_2(t)$, and $e(t) = e_1(t) - e_2(t)$.

Since $e_1(t)$ and $e_2(t)$ are white noise processes, $e(t)$ is also a white noise process.

Therefore, $Y(t)$ must be an AR(1) process.

$Y(t)$ is stationary as long as $|a - b| < 1$.

Since $X_1(t)$ and $X_2(t)$ are themselves $I(1)$, and their difference (i.e., a specific linear combination) is stationary whenever $|a - b| < 1$, the two processes are cointegrated whenever $|a - b| < 1$.

Q7.

(i) The main linear models used for modeling stationary time series are:

- **Autoregressive process (AR)**

  An autoregressive process of order p (the notation AR(p) is commonly used) is a sequence of random variables $\{X_t\}$ defined consecutively by the rule:

  $X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \ldots\ldots.. + \alpha_p (X_{t-p} - \mu) + e_t$

  Thus the autoregressive model attempts to explain the current value of X as a linear combination of past values with some additional externally generated random variation.

- **Moving average process (MA)**

  A moving average process of order q, denoted MA(q), is a sequence {Xt } defined by the rule:

  $X_t = \mu + e_t + \beta_1 e_{t-1} + \ldots\ldots.. + \beta_q e_{t-q}$

The moving average model explains the relationship between the X_t as an indirect effect, arising from the fact that the current value of the process results from the recently past random error terms as well as the current one.

Autoregressive moving average process (ARMA)

The two basic processes (AR and MA) can be combined to give an autoregressive moving average, or ARMA, process. The defining equation of an ARMA(p,q) process is:

$X_t = \mu + \alpha_1 (X_{t-1} - \mu) + \alpha_2 (X_{t-2} - \mu) + \ldots\ldots.. + \alpha_p (X_{t-p} - \mu) + e_t + \beta_1 e_{t-1} + \ldots\ldots.. + \beta_q e_{t-q}$

[6]

(ii)

a. This is MA(1) process and hence it is stationary (as it is the sum of stationary white noise terms). Therefore we can classify it as ARIMA(0,0,1).

b. This is an ARMA(2,3) process.

This process cannot be differenced, so to be able to classify it as an ARIMA(2,0,3), we must check that it is I(0), i.e. stationary.

Since $(1-1.4B^2)\, X_t = \varepsilon_t + 0.5\, \varepsilon_{t-3}$, the characteristic equation of the AR terms is:

$\phi(\lambda) = 1-1.4\, \lambda^2 = 0 \quad \Rightarrow \lambda = \pm 0.8452$

Since both of the roots are less than one in magnitude the process is not stationary and so we cannot classify it as an ARIMA(2,0,3) .

It is a non-stationary ARMA(2,3) process.

c. This is an ARMA(2,1) process.

This process can be differenced as follows:

$X_t - 1.4X_{t-1} + 0.4X_{t-2} = \varepsilon_t + \varepsilon_{t-1}$

$(X_t \,.\, X_{t-1}) - 0.4\,(x_{t-2} - X_{t-2}) = \varepsilon_t + \varepsilon_{t-1}$

$\Delta X_t - 0.4\,\Delta X_{t-1} = \varepsilon_t + \varepsilon_{t-1}$

This process cannot be differenced again, so to be able to classify it as an ARIMA(1,1,1), we must check whether this differenced process is stationary (i.e. the original process is I (1)).

Since $(1- 0.4B)\,\Delta X_t = \varepsilon_t + \varepsilon_{t-1}$, the characteristic equation of the differenced AR terms is:

$\phi(\lambda) = 1 - 0.4\,\lambda = 0 \qquad \Rightarrow \lambda = 2.5$

Since the root is greater than one in magnitude the differenced process is stationary (i.e. the original process is I (1)). Therefore, we can classify it as an ARIMA(1,1,1) .

[4]

(iii) The process is stationary as it the sum of stationary white noise terms, so we can calculate the autocovariance function (ignoring the 3.1's as they will not affect the results and noting that $\gamma_k = \gamma_{-k}$):

$\gamma_0 = \text{Cov}\,(X_t, X_t) = \text{var}(X_t)$
$= \text{Cov}\,(\varepsilon_t + 0.25\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} + 0.25\,\varepsilon_{t-3},\ \varepsilon_t + 0.25\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} + 0.25\,\varepsilon_{t-3})$
$= \sigma^2 + 0.25^2\,\sigma^2 + 0.5^2\,\sigma^2 + 0.25^2\,\sigma^2$
$= 1.375\,\sigma^2$

$\gamma_{+/-1} = \text{Cov}\,(X_t, X_{t-1})$
$= \text{Cov}\,(\varepsilon_t + 0.25\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} + 0.25\,\varepsilon_{t-3},\ \varepsilon_{t-1} + 0.25\,\varepsilon_{t-2} + 0.5\,\varepsilon_{t-3} + 0.25\,\varepsilon_{t-4})$
$= 0.25\,\sigma^2 + (0.5)\,(0.25)\,\sigma^2 + (0.5)\,(0.25)\,\sigma^2$
$= 0.5\,\sigma^2$

$\gamma_{+/-2} = \text{Cov}\,(X_t, X_{t-2})$
$= \text{Cov}\,(\varepsilon_t + 0.25\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} + 0.25\,\varepsilon_{t-3},\ \varepsilon_{t-2} + 0.25\,\varepsilon_{t-3} + 0.5\,\varepsilon_{t-4} + 0.25\,\varepsilon_{t-5})$
$= 0.5\,\sigma^2 + 0.25^2\,\sigma^2$
$= 0.5625\,\sigma^2$

$\gamma_{+/-3} = \text{Cov}\,(X_t, X_{t-3})$
$= \text{Cov}\,(\varepsilon_t + 0.25\,\varepsilon_{t-1} + 0.5\,\varepsilon_{t-2} + 0.25\,\varepsilon_{t-3},\ \varepsilon_{t-3} + 0.25\,\varepsilon_{t-4} + 0.5\,\varepsilon_{t-5} + 0.25\,\varepsilon_{t-6})$
$= 0.25\,\sigma^2$
$\gamma_{+/-k} = 0$ for $|k| > 3$

Since $\rho_k = \gamma_k / \gamma_0$, the autocorrelation function is:

$\rho_0 = 1$
$\rho_{+/-1} = 0.364$

$\rho_{+/-2} = 0.409$
$\rho_{+/-3} = 0.182$
$\rho_{+/-k} = 0$ for $|k| > 3$

[5]

Q8.

(i)

$$X_t = A_t + B_t$$
$$A_t = 0.5A_{t-1} + 0.5B_{t-1} + e_t^{(A)}$$
$$B_t = 0.7A_{t-1} - 0.7A_{t-2} + e_t^{(B)}$$

Using matrix notation we get,

$$Y_t = \begin{pmatrix} A_t \\ B_t \end{pmatrix} = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix}\begin{pmatrix} A_{t-1} \\ B_{t-1} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix}\begin{pmatrix} A_{t-2} \\ B_{t-2} \end{pmatrix} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$$

Or, $Y_t = \begin{pmatrix} 0.5 & 0.5 \\ 0.7 & 0 \end{pmatrix} Y_{t-1} + \begin{pmatrix} 0 & 0 \\ -0.7 & 0 \end{pmatrix} Y_{t-2} + \begin{pmatrix} e_t^{(A)} \\ e_t^{(B)} \end{pmatrix}$

The Eigen values of first matrix are given by the following equation,

$\lambda^2 - 0.5\lambda - 0.35 = 0$

Hence, $|\lambda| < 1$

Similarly for second matrix, $|\lambda| < 1$

As all the Eigen values are less than 1, hence the process Y_t is stationary. **[3]**

(ii)

a) $X_t = (\alpha + 1)X_{t-1} - (\alpha + 0.25\alpha^2)X_{t-2} + 0.25\alpha^2 X_{t-3} + e_t$

Or, $X_t - X_{t-1} = \alpha\,(X_{t-1} - X_{t-2}) - 0.25\alpha^2(X_{t-2} - X_{t-3}) + e_t$

Or, $Y_t = \alpha\, Y_{t-1} - 0.25\alpha^2 Y_{t-2} + e_t,$ $\; assuming\; Y_t = X_t - X_{t-1}$

So X_t is ARIMA(2,1,0) process if it is I(1)

Now, $(1 - \alpha B + 0.25\alpha^2 B^2)\; Y_t = e_t$

The characteristic equation is, $1 - \alpha\lambda + 0.25\alpha^2\lambda^2 \;\; = 0,\; or\; \lambda = \frac{2}{\alpha}$

To meet stationary condition, $|\lambda| > 1, or\; |\alpha| < 2$

Hence X_t is ARIMA(2,1,0) process with $|\alpha| < 2$ **[2]**

b) Now Cov(Y_t , e_t) = Cov(e_t , e_t) = σ^2

$\gamma_0 = Cov(Y_t, Y_t) = \alpha\gamma_1 - 0.25\,\alpha^2\gamma_2 + \sigma^2$ ....(1)

Taking co-variances with Y_{t-1} , Y_{t-2} and Y_{t-k} we ge

$\gamma_1 = \alpha\gamma_0 - 0.25\,\alpha^2\gamma_1$ ....(2)

$\gamma_2 = \alpha\gamma_1 - 0.25\,\alpha^2\gamma_0$ ....(3)

$\gamma_k = \alpha\gamma_{k-1} - 0.25\,\alpha^2\gamma_{k-2}$

From (2), $\gamma_1 = \frac{\alpha\gamma_0}{(1+0.25\,\alpha^2)}$ ....(4)

Substituting (3) in (1) we get,

$\gamma_0 = \alpha\gamma_1 - 0.25\,\alpha^2(\alpha\gamma_1 - 0.25\,\alpha^2\gamma_0) + \sigma^2$

Or, $(1-(0.25\,\alpha^2)^2\,)\gamma_0 = \frac{\alpha(1-0.25\,\alpha^2)\alpha}{1+0.25\,\alpha^2}\gamma_0 + \sigma^2$

Or, $\gamma_0 = \frac{(1+0.25\,\alpha^2)}{(1-0.25\,\alpha^2)^3}\sigma^2$

Hence, $\gamma_1 = \frac{\alpha}{(1+0.25\,\alpha^2)}\frac{(1+0.25\,\alpha^2)}{(1-0.25\,\alpha^2)^3}\sigma^2 = \frac{\alpha}{(1-0.25\,\alpha^2)^3}\sigma^2$

For k ≥ 2, $\gamma_k = \alpha\gamma_{k-1} - 0.25\,\alpha^2\gamma_{k-2}$ ....(5)

Now γ_k follows the below equation

$\lambda_1 + k\lambda_2 = (0.5\alpha)^{-k}\ \gamma_k$ .... (6)

Substituting k by k-1 and k-2 we get

$\lambda_1 + (k-1)\lambda_2 = (0.5\alpha)^{-(k-1)}\ \gamma_{k-1}$

$\lambda_1 + (k-2)\lambda_2 = (0.5\alpha)^{-(k-2)}\ \gamma_{k-2}$

Substituting the above two equation in eqn (5) we get,

$\gamma_k = \alpha(0.5\alpha)^{(k-1)}[\lambda_1 + (k-1)\lambda_2] - 0.25\,\alpha^2(0.5\alpha)^{(k-2)}[\lambda_1 + (k-2)\lambda_2]$

Or, $\gamma_k = \lambda_1\alpha^k\,(0.5)^{(k-1)}\left[1 - \frac{0.25}{0.5}\right] + \lambda_2\alpha^k\,(0.5)^{(k-1)}[(k-1) - \frac{1}{2}(k-2)]$

Or, $\gamma_k = \lambda_1\alpha^k\,(0.5)^k + k\lambda_2\alpha^k\,(0.5)^k$

Or, $\lambda_1 + k\lambda_2 = (0.5\alpha)^{-k}\ \gamma_k$

Which is the original equation. Hence γ_k follows equation (6)

Now putting k = 0 we get,

$$\lambda_1 = \gamma_0 = \frac{(1+0.25\,\alpha^2)}{(1-0.25\,\alpha^2)^3}\sigma^2$$

Putting k = 1 we get,

$$\lambda_2 = \frac{1}{0.5\alpha}\gamma_1 - \lambda_1 = \frac{\sigma^2}{(1-0.25\,\alpha^2)^2}$$ **[11]**

c) α = 0.04, hence $Y_t = 0.04\,Y_{t-1} - 0.0004Y_{t-2} + e_t$

$Y_t = X_t - X_{t-1} \quad Or \quad X_t = Y_t + X_{t-1}$

Since $x_1, x_2, \ldots\ldots, x_{50}$ are observed values

$X_{51} = Y_{51} + X_{50}$

$X_{52} = Y_{52} + X_{51}$

So the forecasted values are

$x_{51} = y_{51} + x_{50}$ and $x_{52} = y_{52} + x_{51}$

Where

$y_{51} = 0.04\,(x_{50} - x_{49}) - 0.0004(x_{49} - x_{48})$

And $y_{52} = 0.04\,y_{51} - 0.0004(x_{50} - x_{49})$ **[2]**

Q9.

i)
Since e_t are independent from X_t, X_{t-1}, … and $E(e_t) = 0$ we have that

$$E(X_t) = \mu + E(e_t\sqrt{\alpha + \beta(X_{t-1} - \mu)^2})$$
$$E(X_t) = \mu + E(e_t)E(\sqrt{\alpha + \beta(X_{t-1} - \mu)^2})$$
$$E(X_t) = \mu + 0 * E(\sqrt{\alpha + \beta(X_{t-1} - \mu)^2})$$
$$E(X_t) = \mu$$

$$Cov(X_{t,}X_{t-s}) = E(X_t X_{t-s}) - E(X_t)E(X_{t-s})$$

$$Cov(X_{t,}X_{t-s}) = E(\left(\mu + e_t\sqrt{\alpha + \beta(X_{t-1} - \mu)^2}\right)\left(\mu + e_{t-s}\sqrt{\alpha + \beta(X_{t-s-1} - \mu)^2}\right)) - \mu^2$$

Let $A_t = \sqrt{\alpha + \beta(X_{t-1} - \mu)^2}$
$$Cov(X_{t,}X_{t-s}) = E((\mu + e_tA_t)(\mu + e_{t-s}A_{t-s})) - \mu^2$$

$$Cov(X_{t,}X_{t-s}) = E(\mu^2 + \mu e^t A_t + \mu e^{t-s}A_{t-s} + e_t e_{t-s}A_tA_{t-s}) - \mu^2$$

$$Cov(X_{t,}X_{t-s}) = E(\mu^2) + \mu E(e^t)E(A_{t)} + \mu E(e^{t-s})(A_{t-s)} + E(e_t e_{t-s}A_tA_{t-s}) - \mu^2$$
$$Cov(X_{t,}X_{t-s}) = \mu^2 + \mu * 0 * E(A_{t)} + \mu * 0 * (A_{t-s)} + E(e_t e_{t-s}A_tA_{t-s}) - \mu^2$$
$$Cov(X_{t,}X_{t-s}) = E(e_t e_{t-s}A_tA_{t-s})$$

Now e_t is independent of X_{t-1} as above, and that e_t is independent of $e_{t-s}A_tA_{t-s}$
$$Cov(X_{t,}X_{t-s}) = E(e_t)E(e_{t-s}A_tA_{t-s})$$
$$Cov(X_{t,}X_{t-s}) = 0 * E(e_{t-s}A_tA_{t-s})$$
$$Cov(X_{t,}X_{t-s}) = 0$$

Thus $X_{t,} and\ X_{t-s}$ are uncorrelated.

Q10.

i) p=1 , q =4 hence it will follow ARMA(1,4)

[1]

ii) Non-linear non stationary time series models includes:-

Bilinear models are those that exhibit "bursty" behaviour:

$$X_n - \alpha(X_{n-1} - \mu) = \mu + e_n + \beta e_{n-1} + b(X_{n-1} - \mu)e_{n-1}$$

Threshold autoregressive models are used to model "cyclical" behaviour:

$$X_n = \mu + \begin{cases} \alpha_1(X_{n-1} - \mu) + e_n \; if X_{n-1} \le d, \\ \alpha_2(X_{n-1} - \mu) + e_n \; if X_{n-1} > d, \end{cases}$$

Random coefficient, autoregressive models is a sequence of independent random variables:

$X_t = \mu + \alpha_t(X_{t-1} - \mu) + e_t, where \{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ is a sequence of independent random variables

Autoregressive with conditional heteroscedasticity (ARCH) models are used to model asset prices, where we require the volatility to depend on the size of the previous value:

$$X_t = \mu + e_t \sqrt{\alpha_0 + \sum_{k=1}^{p} \alpha_k (X_{t-k} - \mu)^2}$$

..Only 4 names (0.25*4=1.0) + Definitions (4*0.5=2)

[3]

iii) X_t follows MA(1), hence we can write

$X_t = e_t + \beta e_{t-1}$, Where $e_t \sim (0, \sigma^2)$

Now, $\text{var}(X_t) = \text{var}(e_t + \beta e_{t-1})$

$= \text{var}(e_t) + \beta^2 \text{var}(e_{t-1})$

$$=(1+\beta^2)\sigma^2 \text{ ....(1)}$$

Now, $\Delta Y_t = (0.6+0.3t+X_t)-[0.6+0.3(t-1)+X_{t-1}]$

$$=0.3+(X_t - X_{t-1})$$

Hence $\text{var}(\Delta Y_t)=[\text{cov}(X_t - X_{t-1}, X_t - X_{t-1})]$

$$=[2\gamma_X(0)-\gamma_X(-1)-\gamma_X(1)] \text{ ....(2)}$$

$$\therefore Now, \gamma_x(0)=(1+\beta^2)\sigma^2$$

And, $\gamma_x(1)=\gamma_x(-1)=\text{cov}(e_t+\beta e_{t-1}, e_t+\beta e_{t-1})=\beta\sigma^2$

Therefore, from (2) we get,

$$\text{var}(\Delta Y_t)=[2(1+\beta^2)\sigma^2-2\beta\sigma^2]=2[1-\beta+\beta^2]\sigma^2$$

..(0.5)

Now, $\text{var}(\Delta Y_t)-\text{var}(X_t)$

$$=[2-2\beta+2\beta^2]\sigma^2-(1+\beta^2)\sigma^2$$

$$=[1-2\beta+\beta^2]\sigma^2$$

$$=(1-\beta^2)\sigma^2>0$$

Hence the standard deviation of first difference of Y_t is higher than that of X_t

[6]

Q11.

Solution 10:

i)

$$
\begin{aligned}
y_t &= \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \epsilon_t \\
\mathrm{Cov}(y_t, y_{t-1}) &= \mathrm{Cov}(\alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \epsilon_t, y_{t-1}) \\
\gamma_1 &= \alpha_1 \gamma_0 + \alpha_2 \gamma_1 + \alpha_3 \gamma_2 + 0
\end{aligned}
$$

Dividing both sides by γ_0 gives autocorrelation on LHS at lag 1

$$p_1 = \alpha_1 + \alpha_2 p_1 + \alpha_3 p_2 - - - - - - - - (1)$$

$$\Rightarrow p_1(1-\alpha_2) = \alpha_1 + \alpha_3 \rho_2$$

$$\mathrm{cov}(y_t, y_{t-2}) = \mathrm{cov}(\alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \alpha_3 y_{t-3} + \epsilon_t, y_{t-2})$$

$$\gamma_2 = \alpha_1 \gamma_1 + \alpha_2 \gamma_0 + \alpha_3 \gamma_1 + 0$$

Dividing both sides by γ_0

$$p_2 = \alpha_1 p_1 + \alpha_2 + \alpha_3 p_1$$

$$\Rightarrow \quad p_2 = (\alpha_1 + \alpha_3) p_1 + \alpha_2 - - - - - - - - (2)$$

Substituting (2) in (1)

$$
\begin{aligned}
p_1(1-\alpha_2) &= \alpha_1 + \alpha_3[(\alpha_1 + \alpha_3) p_1 + \alpha_2] \\
&= \alpha_1 + \alpha_3(\alpha_1 + \alpha_3) p_1 + \alpha_3 \alpha_2
\end{aligned}
$$

$$\rho_1(1 - \alpha_2 - \alpha_3(\alpha_1 + \alpha_3)) = \frac{\alpha_1 + \alpha_3 \alpha_2}{\rho_1} = \frac{\alpha_1 + \alpha_3 \alpha_2}{1 - \alpha_2 - \alpha_3(\alpha_1 + \alpha_3)}$$

$$= \left. \frac{(\alpha_1 + \alpha_3)[\alpha_1 + \alpha_2 \alpha_3]}{1 - \alpha_2 - \alpha_3(\alpha_1 + \alpha_3)} \right] + \alpha_2$$

ii)

$$\rho_1 = \frac{0.3 + 0.09}{1 - 0.3 - 0.3(06)}$$

$$\frac{0.39}{0.52} = 0.75$$

$$
\begin{aligned}
p_2 &= 0.6 p_1 + \alpha_2 \\
&= 0.6(075) + 0.3 \\
&= 0.45 + 0.3 \\
&= 0.75
\end{aligned}
$$

(2)

iii) PACF at Lag' $= ACF$ at Lag $' = 0.75$

PACF at Lag2 $= \frac{p_2 - p_1^2}{1 - p_1^2} = \frac{0.75 - 0.75^2}{1 - 0.75^2} = 0.428571$ (2)

iv) The process is a AR(3) process and hence the PACF will be significant for lags up to 3. The PACF for Lags 1 and 2 is 0.75 and 0.428 are significant. From Lag 4 onwards, the PACF values will become insignificant (1.5 Marks)

There is no MA order in the process. In an AR process, the ACF gradually falls to zero. Here ACF at lag 1 and lag 2 are 0.75 but they will gradually fade off after a few more lags. Higher- (3)

lag autocorrelations will satisfy the Yule-Walker equation $\rho_k = 0.3\ (\rho_{k-1} + \rho_{k-2} + \rho_{k-3})$. So the values will tail off quite quickly to zero, always taking positive values. (1.5 Marks)

Q12.

i) Lack of stationarity:

- Lack of stationarity may be caused by the presence of deterministic effects in the quanitity being observed. For ex. Deterministic trend or cycle such as seasonal effects.
- If the process observed is the integrated version of a more fundamental process.
- For ex. A company which sells greeting cards will find that the sales in some months will be much higher than in others. [3]

ii)

a) Any of the below is a form of ARIMA(p,d,q) process

$(1-B)^d\ \phi(B)\ (X_t-\mu) = \theta(B)e_t$ where $\phi(B) = 1-\sum_1^p B^i \alpha i$ and $\theta(B) = \sum_1^q B^j \beta j$

$(\nabla^d Xt - \mu) = \sum_1^p \alpha i(\nabla^d Xt - 1 - \mu) + e_t + \sum_1^q B^j et - j$ [2]

b) Main steps involved in Box-Jenkins methodology:

- Tentative identification of a model from the ARIMA class
- Estimation of the parameter in the identified model
- Diagnostic checks [2]

c) If the sample auto correlation coefficients decay slowly from 1 then this indicates that further differencing is required. This is not the case for d=1, which means that the differencing of the original series is required once hence d=1.
Further the correct value of d minimise the sample variance. This also indicates that d=1. [2]

d) Classify the time series as ARIMA(p,d,q)

1. $X_t = 0.8e_{t-1} + e_t$

This is a MA(1) process and hence it is stationary. Therefore we can calssify it as ARIMA(0,0,1) process.

2. $Xt=2X_{t-2}+ e_t +0.5\ e_{t-3}$

This is an ARMA(2,3) process. This process can not be differenced so we can classify it as ARIMA (2,0,3), we also must see if I(0) is stationary.

$=(1-2B^2)X_t = e_t+0{,}5e_{t-3}$, the characeteristic equation of AR terms is
$= \phi(\lambda)=1-2\lambda^2=0$
λ = +- 1/√2, both the roots of the characteristic equation are less than 1. We can not classify the process as ARIMA(2,0,3). Hence its a non stationary ARMA(2,3) process.

3. $Xt=1.5X_{t-1}+.5X_{t-2}+ e_t + e_{t-1}$

This is an ARMA(2,1) process.
The process can be differenced as follows
$X_t-1.5X_{t-1}+.5X_{t-2} = e_t + e_{t-1}$
$= (X_t-X_{t-1})-.5(X_{t-1}-X_{t-2}) = e_t + e_{t-1}$
$=\nabla Xt - .5\nabla Xt - 1 = e_t + e_{t-1}$

This process can not be differenced further hence we take d=1.
To classify the process as ARIMA(1,1,1) we need to check if the differenced process is stationary.
The characteristic equation is $(1-.5\lambda)=0$ which implies $\lambda=2$
The differenced process is stationary hence we can classify it as ARIMA(1,1,1) process.
[6]

Q13.

(i) A stochastic process is weakly stationary if it has constant mean and the covariance is constant for each fixed lag.

(ii) The moving average process is $X_n = Z_n + \beta Z_{n-1}$.

The mean of the process is $E[X_n] = (1 + \beta)\,\mu$. This is constant.

The variance:

$$\begin{aligned} \mathrm{Var}(X_n) &= \mathrm{Var}(Z_n + \beta Z_{n-1}) \\ &= (1+\beta^2)\,\sigma^2. \end{aligned}$$

The covariance

$$\begin{aligned} \mathrm{Cov}(X_n, X_{n+1}) &= \mathrm{Cov}(Z_n + \beta Z_{n-1}, Z_{n+1} + \beta Z_n) \\ &= \beta\sigma^2. \end{aligned}$$

For $m > 1$, $\mathrm{Cov}(X_n, X_{n+m}) = \mathrm{Cov}(Z_n + \beta Z_{n-1}, Z_{n+m} + \beta Z_{n+m-1})$
$= 0.$

The covariance at higher lags are 0 since there is no overlap between the Z's. The covariance at negative lags are the same as those at the corresponding positive lags. Since none of these expressions depends on n, it follows that the process is weakly stationary.

(iii) Write the model equation as

$(1 - 1.5B + .5B^2)X = Z,$

where B is the back-shift operator. The polynomial in B factorizes as

$(1 - B)(1 - .5B),$

Since one of the roots of the polynomial has magnitude 1, the process is NOT stationary.

(iv) The process X is ARIMA(1,1,0), so $(1-B)X$ is AR(1). Define the process Y as $Y = (1 - B)\,X$, and write this AR(1) process as

$(1 - .5B)Y = Z$.

According to standard formulae, the autocorrelation at lag 1 is 0.5.