Lecture 1

INSTITUTE OF ACTUARIAL
& QUANTITATIVE STUDIES

Class: TY BSc

Subject : Statistical and Risk Modelling-4

Subject Code: PUSAQF605A

Chapter: Unit 1

Chapter Name :- Copulas

Archimedean Copulas

General definition of Archimedean copulas

Copulas in the Archimedean family are of the form:

$$C[u_1,\ldots,u_d] = \psi^{[-1]}\left(\sum_{i=1}^{d}\psi(u_i)\right)$$

In order to be valid, **the generator function** $\psi:[0,1]\rightarrow[0,\infty]$ **must be a continuous, strictly decreasing, convex function with** $\psi(1)=0$.

Today's Agenda

1. Introduction

2. Marginal and joint distributions

3. Terminology of associations

4. <u>Copulas</u>
    1. Definition
    2. Three properties

5. Sklar's theorem

6. Tail dependence and the survival copula

Today's Agenda

7. Fundamental copulas
    1. Independence (or product) copula
    2. Co-monotonic (or minimum) copula
    3. Counter-monotonic (or maximum) copula
    4. The multivariate case

8. Explicit copulas
    5. Gumbel copula
    6. Clayton copula
    7. Frank copula
    8. Archimedean copulas

Today's Agenda

9. Implicit copulas
   1. Gaussian copula
   2. Student's t copula

10. Choosing and fitting a suitable copula function

11. Calculating probabilities using copula

1. Introduction

Aim → To be able to study joint probabilities

Joint Probabilities of Events

Example 1. Default on multiple types of loans
2. Loss on 2 different types of insurance policies

Example Home → X_H
Motor → X_M
$F_{X_M,X_H}(3,2) = ?$

- There is interdependency and loans default and losses in insurance.

1 Introduction Contd

Problem with Joint Probabilities

- Joint pdf's make it hard to model the nature of relationship between two variables

And hence as an alternative we use Copulas

- Copulas
  It's a function that takes marginal cdf's as input & gives joint cdf as output

  Marginal PDF's → Copulas → Joint CDF
  Copulas → function of marginal cdf's
  $C[F_x(x), F_y(y)] = F_{xy}(xy)$
  based on relationship between two variables

2. Marginal & Joint distributions

Joint cdf:

Deriving the joint CDF by integrating the joint PDF :

Computing the marginal PDF from a joint PDF by integrating out the other variable:

Computing a marginal CDF by integrating the marginal PDF:

2. Marginal & Joint distributions

The joint PDF for two continuous random variables X and Y is:

$$f_{X,Y}(x,y)=\frac{1}{20}(x+4y),\quad 0<x\le 2,\ 0<y\le 2$$

(i) Derive a formula for the joint CDF, $F_{X,Y}(x,y)$.

(ii) Derive formulae for the marginal PDFs, $f_X(x)$ and $f_Y(y)$, and comment on whether X and Y are independent.

(iii) Derive formulae for the marginal CDFs, $F_X(x)$ and $F_Y(y)$.

3. Terminology of associations

Types of associations:

- Pearson's linear correlation coefficient

- Rank correlation coefficients

- Tail dependence

Coefficient of upper tail dependence -

Coefficient of lower tail dependence -

4. Definition of Copulas

A copula is a function that expresses a multivariate cumulative distribution function in terms of the individual marginal cumulative distributions.

For a bivariate distribution, the copula is a function C_{XY} defined by:

$$C_{XY}\left[F_X(x), F_Y(y)\right] = P(X \le x, Y \le y) = F_{X,Y}(x,y)$$

This is often written in the more compact form:

$$C[u,v] = F_{X,Y}(x,y) \quad \text{where } u = F_X(x) \text{ and } v = F_Y(y)$$

This definition can be extended to the multivariate case where we have:

$$C[u_1, u_2, \ldots, u_d] = F_{X_1, X_2, \ldots, X_d}(x_1, x_2, \ldots, x_d) \quad \text{where } u_i = F_{X_i}(x_i)$$

4. Properties of Copulas

- $C(u_1, u_2, u_3 \dots u_i^*, \dots u_d) > C(u_1, u_2, \dots, u_i, \dots)$ if $u_i^* > u_i$
  Copulas is an increasing fn of inputs

- $C(1,1,1,u,1,1) = u$

- $C(u_1, u_2, u_3 \dots u) \in (0,1)$
  copulas always returns a valid probability

5. Sklar's Theorem

- Whenever you have a joint distribution and corresponding marginal distribution you can always express joint CDF as a function of marginal cdf's using a copula

 For a given joint distribution and corresponding marginal distribution , there exists a unique copula which links them & vice versa

5. Sklar's Theorem

The joint probability density function for two continuous random variables X and Y is:

$$f_{X,Y}(x,y) = \frac{1}{20}(x+4y), \quad 0 < x \le 2, \ 0 < y \le 2$$

(i) Derive formulae for the inverse cumulative distribution functions $F_X^{-1}(u)$ and $F_Y^{-1}(v)$.

(ii) Hence derive a formula for the copula function $C(u,v) = F_{XY}(x,y)$.

Coefficient of Tail Dependance

5. Survival Formula

6. Coefficient of Upper Tail Dependancy

$$\lambda_\mu = \lim_{\mu \to -1} P(X \geq F_x^{-1}(\mu) \,| Y \geq F_y^{-1}(\mu)$$

Product Copulas

X and Y are independent of each other

C(u,v) = u*v

$$\lambda_L = \lim_{\mu \to 0^+} \frac{C(u,u)}{u} = \lim_{\mu \to 0^+} \frac{u*u}{u} = 0$$

No Correlation
No Lower Tail Dependency 2u

$$\lambda_U = \lim_{\mu \to 1} \frac{1-2u+C(u,u)}{1-\mu} = \frac{1-2\mu+\mu^2}{1-\mu}$$

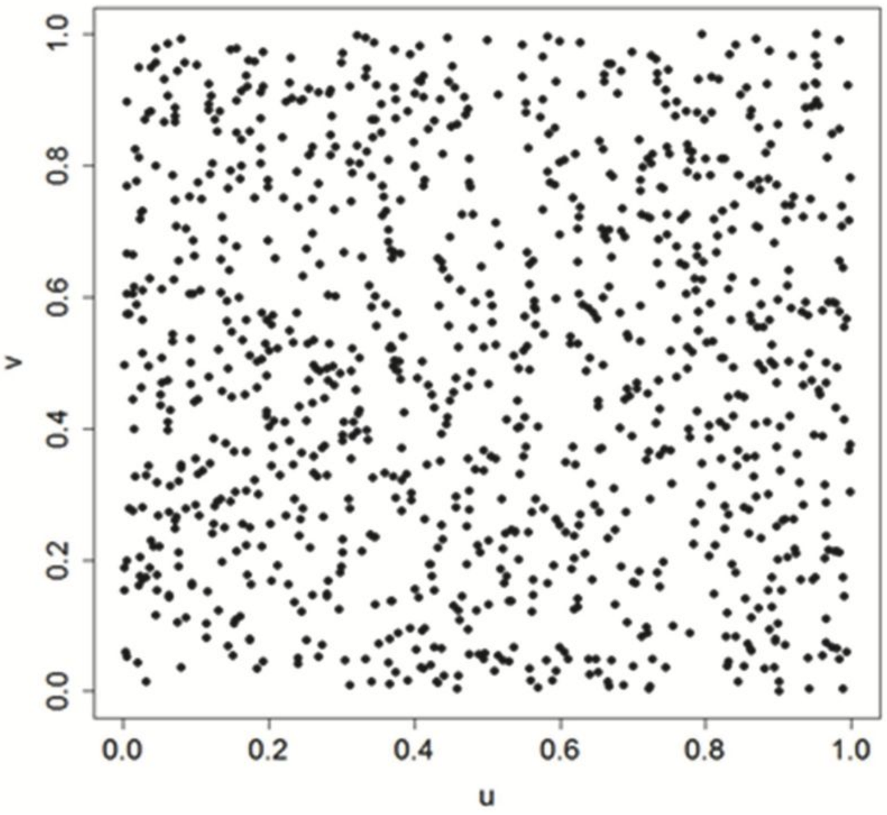
$$= \lim_{\mu \to 1^-} 1 - \mu$$

= 0

No upper tail dependency

7.1 Product Copulas

Scatterplot – Product (Independence) copula



7.2 Co Monotonic Copulas

X and Y have perfect positive interdependence

C(u,v) = min(u,v)

$$C(F_x(x), F_y(y)) = \min\left(F_x(x), F_y(y)\right)$$

Ex.- $Y = X + 0.01$

$$\begin{aligned} P(X \le x, Y \le y\,) &= P(X \le x\,, X + 0.01 \le y) \\ &= P(X \le x\,, X \le y - 0.01\,) \\ &= P(X \le \min(x, y - 0.01) \\ &= \min(P(X \le x), P(X \le y - 0.01) \\ &= \min(P(X \le x), P(Y \le y) \end{aligned}$$

$$\lambda_L = \lim_{\mu - 0^+} \frac{C(u,u)}{u} = 0$$

$$\lambda_U = \lim_{\mu \to 1} \frac{1-2u+min(u,v)}{1-\mu} = \frac{1-u}{1-u} = 1$$

7.2 Co Monotonic Copulas

X and Y have perfect positive interdependence

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$C(F_x(x), F_y(y)) = \min\left(F_x(x), F_y(y)\right)$

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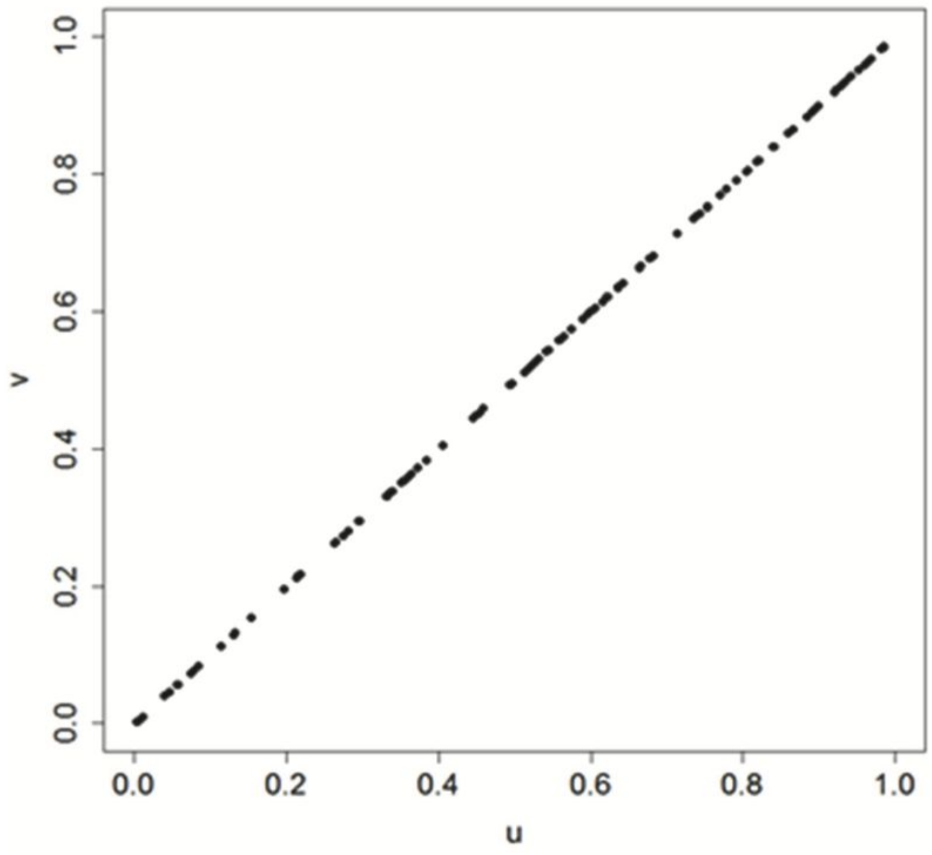
$$\lambda_L = \lim_{\mu - 0^+} \frac{C(u,u)}{u} = 0$$

$$\lambda_U = \lim_{\mu \to 1} \frac{1-2u+min(u,v)}{1-\mu} = \frac{1-u}{1-u} = 1$$

Co Monotonic Copulas

Scatterplot – Co-monotonic copula



7.3 Counter Monotonic (Maxima) Copulas

X and Y have perfect negative interdependence

C(u,v) = max(u+v-1,0)

$C(F_x(x), F_y(y)) = \max(F_x(x) + F_y(y) - 1, 0)$

- $P(X \leq x, Y \leq y)$
  $= P(X \leq x, -X \leq y)$
  $= P(X \leq x, X \geq y)$
  $= P(-y \leq X \leq x)$
  $= P(X \leq x) - P(X \leq -y)$
  $= P(X \leq x) - P(Y > y)$
  $= P(X \leq x) + P(Y \leq y) - 1$
  Hence Proved

7.3 Counter Monotonic (Maxima) Copulas

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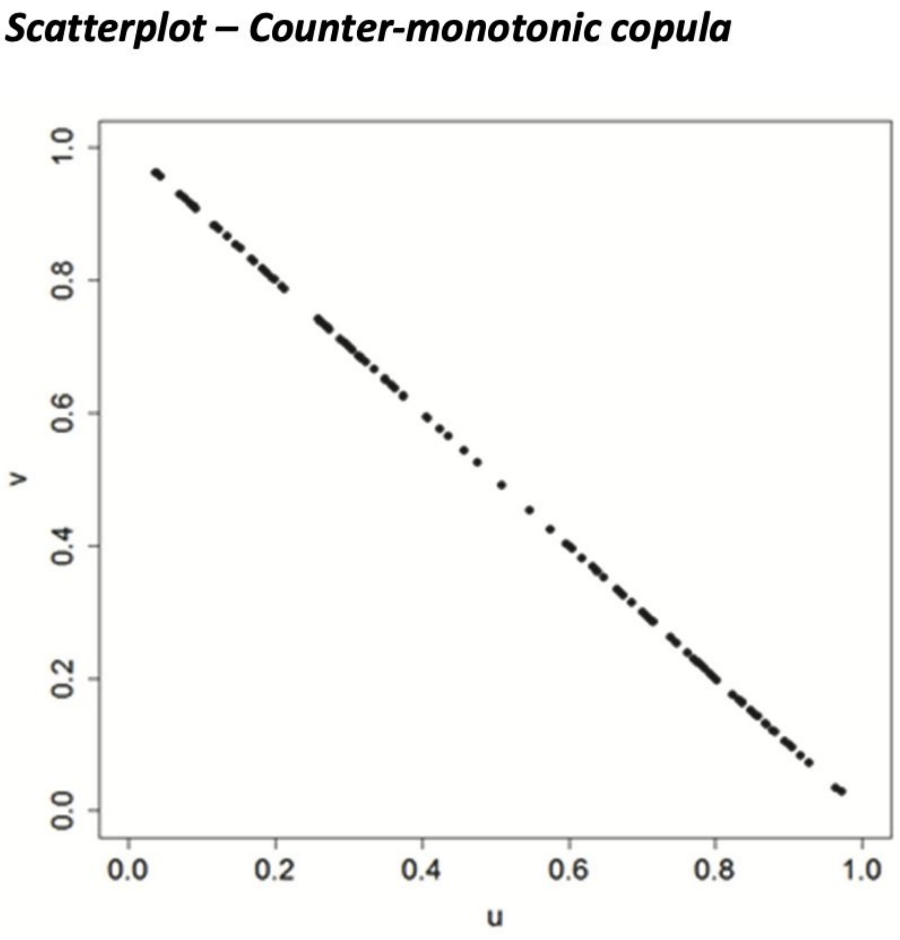
  Hence Proved

7.3 Counter Monotonic (Maxima) Copulas



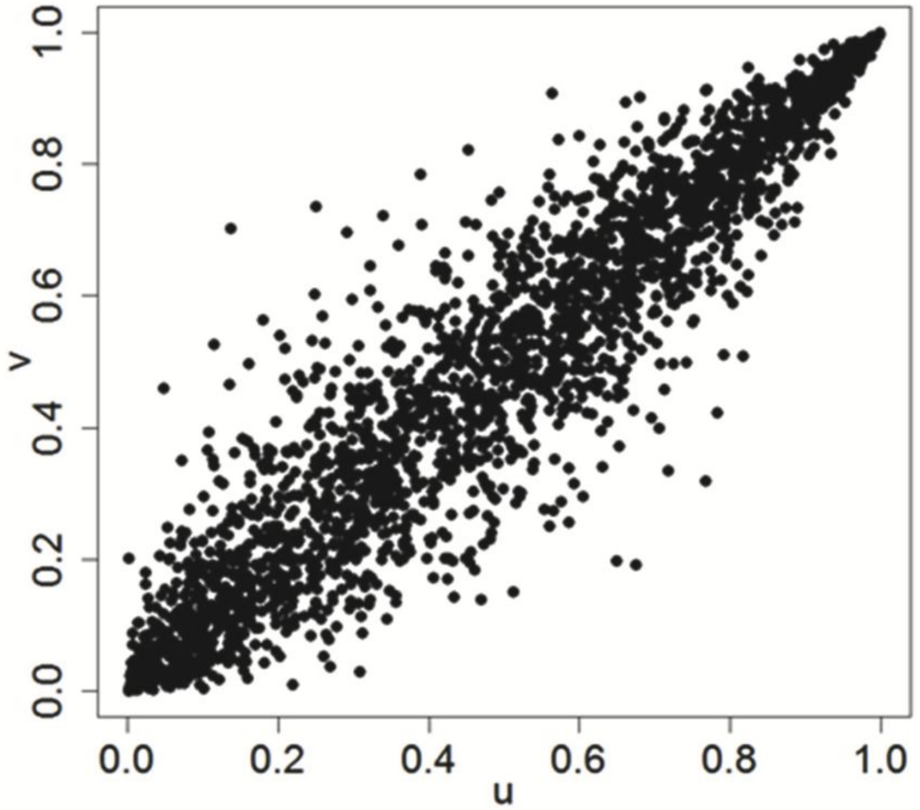
8.1 Gumbel Copulas

The *Gumbel copula* is defined in the bivariate case as:

$$\boldsymbol{C}[\boldsymbol{u},\boldsymbol{v}] = \mathbf{exp}\left\{-\left((-\mathbf{ln}\,\boldsymbol{u})^{\alpha} + (-\mathbf{ln}\,\boldsymbol{v})^{\alpha}\right)^{1/\alpha}\right\}$$

- Gumbel copula describes an interdependence structure in which there is upper tail dependence (the level of which is determined by the parameter α), but there is no lower tail dependence.

8.1 Gumbel Copulas



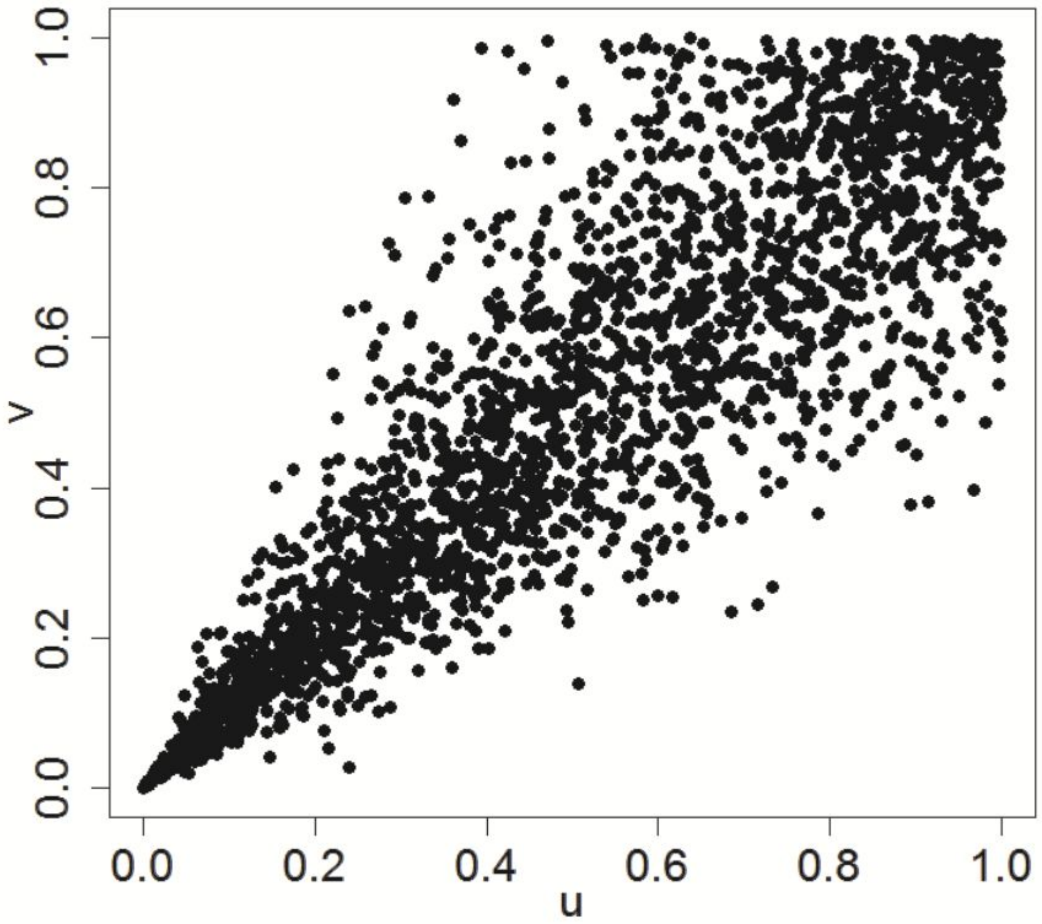
8,2 Clayton Copulas

The *Clayton copula* is defined in the bivariate case as:

$$C[u,v] = \left(u^{-\alpha} + v^{-\alpha} - 1\right)^{-1/\alpha}$$

- Clayton copula describes an interdependence structure in which there is lower tail dependence (the level of which is determined by the parameter α), but there is no upper tail dependence.

8.2 Clayton Copulas



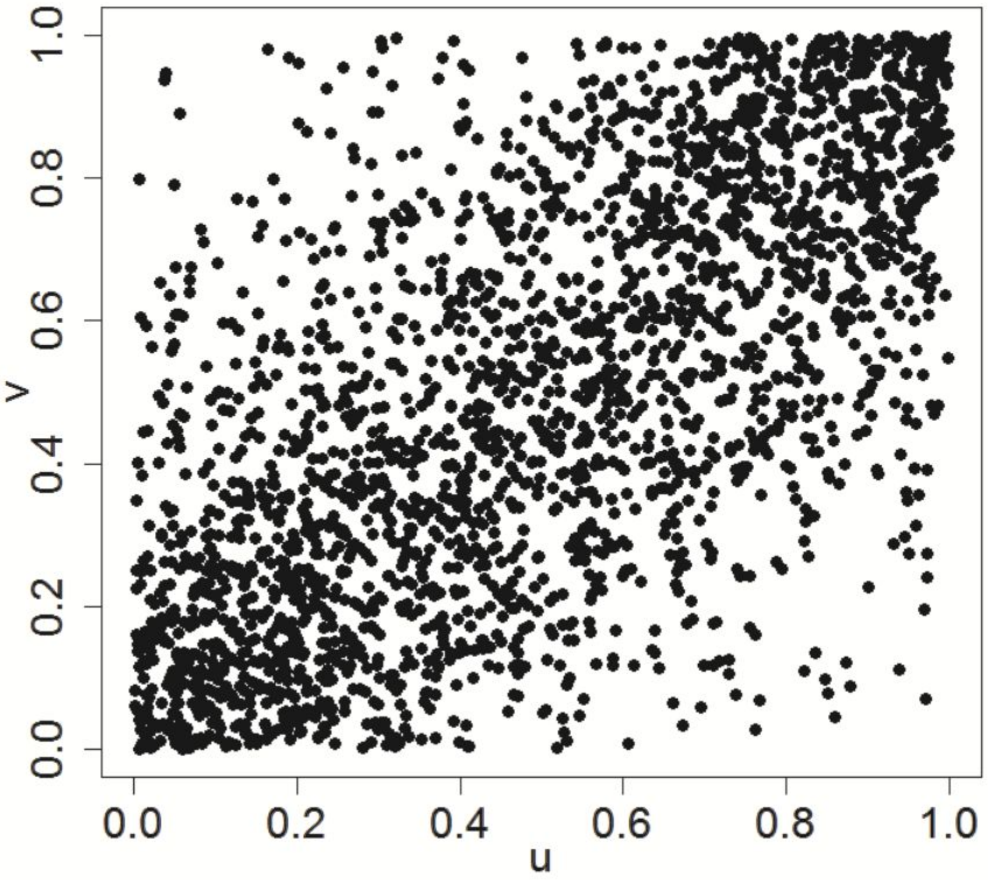
8.3 Frank Copulas

The *Frank copula* is defined in the bivariate case as:

$$C[u,v] = -\frac{1}{\alpha}\ln\left(1+\frac{\left(e^{-\alpha u}-1\right)\left(e^{-\alpha v}-1\right)}{\left(e^{-\alpha}-1\right)}\right)$$

- The Frank copula describes an interdependence structure in which there is no upper or lower tail dependence.

8.3 Frank Copulas



8.4 Archimedean Copulas

Archimedean copulas are described by reference to a *generator function*. In the bivariate case, they take the form:

$$C[u,v]=\psi^{[-1]}\left(\psi(u)+\psi(v)\right)$$

$\psi(x)$ is a strictly decreasing, continuous function of x and $\psi^{[-1]}(x)$ is the pseudo-inverse function

Archimedean copulas are a subset of explicit copulas

8.4 Archimedean Copulas

Pseudo-inverse functions

$$\psi^{[-1]}(x) = \begin{cases} \psi^{-1}(x) & \text{if } 0 \le x \le \psi(0) \\ 0 & \text{if } \psi(0) < x \le \infty \end{cases}$$

where $\psi^{-1}(x)$ denotes the ordinary inverse function obtained by inverting the equation $x = \psi(y)$ to express y in terms of x.

8.4 Archimedean Copulas

Intuition behind Archimedean copulas

Using the definition, we can understand that we are:

- taking probabilities between 0 and 1 (the u_i's or marginal CDFs)
- converting these to numbers greater than 0 using the generator function ψ
- summing the results
- converting the result back to a probability (*ie* the joint CDF) using the inverse function ψ^{-1}.

8.4 Archimedean Copulas

Q For the following generator functions, determine the Archimedean copula and identify the resulting copula

$$\psi(t) = (-\ln t)^{\alpha} \quad \text{where} \quad 1 \le \alpha < \infty$$

$$\psi(t) = -\ln t$$

8.4 Archimedean Copulas

Q

Southwest Re is a start-up reinsurance company that is assessing its economic capital using a Value at Risk approach calibrated to the 95th percentile loss over one year. During its first year, Southwest Re underwrote four excess of loss reinsurance treaties with the following features:

	Excess	***Probability of no loss occurring (ie below excess)***
Cornwall Insurance	£50m	0.995
Devon Insurance	£50m	0.985
Somerset Insurance	£50m	0.975
Dorset Insurance	£50m	0.965

Claims on the reinsurance treaties are assumed to be linked by a Gumbel copula with parameter $\alpha = 2.5$.

8.4 Archimedean Copulas

Continued

The generator function for a Gumbel copula with parameter α is:

$$_{Gu}\psi_{\alpha}\left(F(x)\right) = \left[-\ln(F(x))\right]^{\alpha}$$

The Chief Capital Officer has suggested that, because the probability of no losses occurring on the four reinsurance treaties is greater than 95%, the reinsurer does not need to hold any economic capital.

Verify the Chief Capital Officer's claim that the probability of no losses occurring on the four reinsurance treaties is greater than 95%. [4]

8.4 Archimedean Copulas

Q

An investment company is analysing the likelihood of two corporate bonds defaulting and is trying to decide which copula to use to model their dependence structure.

Bond A has a probability of default in the following year of 0.05.

Bond B has a probability of default in the following year of 0.15.

8.4 Archimedean Copulas

Continued

You are given the following generator functions:

Gumbel copula: $_{Gu}\psi_{\alpha}(F(x)) = (-\ln(F(x)))^{\alpha}$

Clayton copula: $_{Cl}\psi_{\alpha}\left(F(x)\right) = \frac{1}{\alpha}\left(F(x)^{-\alpha} - 1\right)$

(i) Calculate the probability of both bonds defaulting in the following year using:

(a) a Gumbel copula with parameter $\alpha = 2$

(b) a Clayton copula with parameter $\alpha = 2$. [4]

(ii) Explain which copula would be more appropriate. [2]

[Total 6]

9.1 Gaussian Copulas

9.2 Students T Copulas

10. Choosing and fitting a suitable copula function

Examination of the form and levels of association between the variables of interest allows us to select a suitable candidate copula from the list of established copulas or to develop a new bespoke copula.

Different copulas result in different levels of tail dependence.

Choosing and fitting a suitable copula function

We can summarise the upper and lower tail dependence results in the table below:

Copula name	$_L\lambda$	$_U\lambda$
Independence	0	
Co-monotonic	1	
Counter-monotonic	0	
Gumbel	0	$2-2^{1/\alpha}$
Clayton	$2^{-1/\alpha}$ if $\alpha>0$ 0 if $\alpha\leq 0$	0
Frank	0	
Gaussian	0 if $\rho<1$ 1 if $\rho=1$	
Student's *t*	>0 if $\gamma<\infty$, increasing as γ decreases 0 if $\gamma=\infty$ and $\rho\neq 1$ 1 for all γ when $\rho=1$	

Choosing and fitting a suitable copula function

Question

State an appropriate copula to use if the data exhibit the following features:

(a) independence

(b) high upper tail dependence, but no lower tail dependence

(c) a high degree of positive interdependence throughout

(d) high lower tail dependence, but no upper tail dependence

(e) a high degree of negative interdependence throughout

(f) no upper or lower tail dependence

(g) both upper and lower tail dependence.

Calculating probabilities using copulas

Question

Let $X =$ a person's height measured in *cm*, and $Y =$ weight measured in *kg*. Heights and weights are each assumed to be normally distributed, and:

$$P(X \le 180) = 0.81594 \quad \text{and} \quad P(Y \le 70) = 0.69146$$

(i) Calculate the joint probability that a person's height is less than or equal to 180*cm* and that their weight is less than or equal to 70*kg* using:

(a) the independence (or product) copula

(b) the Gaussian copula with $\rho = 0$.

The following table is required for (i)(a). It shows an excerpt of values from the bivariate standard normal cumulative distribution function: $\Phi_{\rho=0}(x,y)$.