INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: B.Sc. Sem 6

Subject : Statistical and Risk Modelling-4

Chapter: Unit 3 (Entire)

Agenda

1. Introduction to Stochastic Processes and its Main Characteristics
    1. Univariate Time Series Processes
    2. Stationarity
    3. Purely Indeterministic Processes
    4. White Noise Process
    5. Auto Covariance and Auto Correlation
    6. PACF

2. Time Series Processes
    1. Auto Regressive (AR) Process
    2. Moving Average Process
    3. ARMA Process
    4. ARIMA Process

3. Markov Process

1 Introduction

Time Series is a Stochastic Process indexed in

- Discrete Time Domain
- Continuous State Space

Time series data is a collection of observations obtained through <u>repeated measurements over time</u>

Example

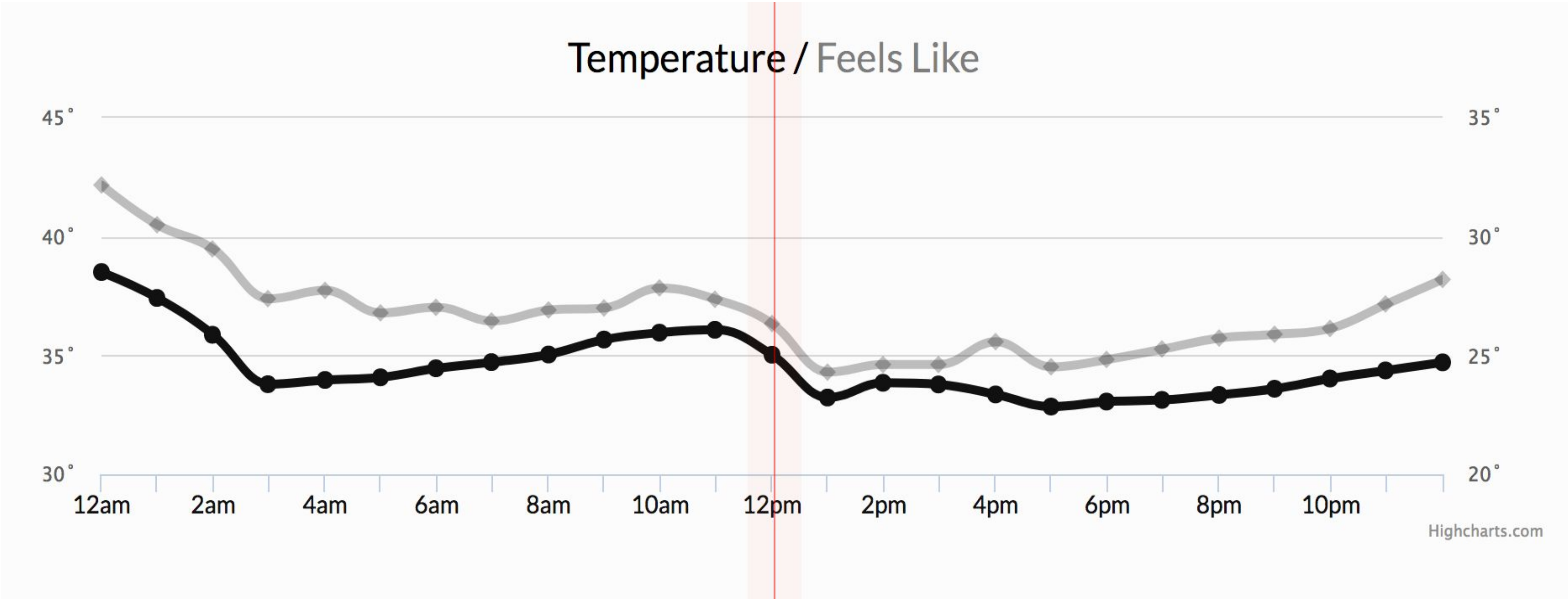
- Closing Price of a Share
- Inflation Rate every quarter
- Temperature on a given day

1 Motivation

Why do we need to model Time Series ?

- To help understand the process better
- To be able to forecast / predict future behaviour
- To improve decision making

1 Plot of a Time Series Process



1.1 Univariate Time Series Process

- Observe a single process at a sequence of different times
- Continuous State Space
- Discrete Time Set

- The term "univariate time series" refers to a time series that consists of **single (scalar) observations recorded sequentially over equal time increments**. ... If the data are equi-spaced, the time variable, or index, does not need to be explicitly given.

1.2 Stationarity

- Stationarity means that statistical properties of the stochastic process remain unchanged.
- Stationarities can be of two types
  - Strong / Strictly Stationary
  - Weak Stationary

A process {Xt} is

Strict Stationarity :- If the joint Distribution of $(X_{t1}, X_{t2}, X_{t3} \ldots X_{tn})$ & $(X_{t1+k}, X_{t2+k} \ldots X_{tn+k})$ **is same i. e. all statistical properties remain the same.**

Weak Stationarity if,

- Mean i.e. E(X_t) is constant
- Covariance depends only on the time lag
- As covariance depends only on the time lag it implies that the variance of the process V(X_t) (Covariance with lag 0) is constant

1.3 Purely Indeterministic Process

The process {Xt} is a purely indeterministic process if knowledge of X_1 ,X_2,X_3 is progressively less useful at predicting the value of X_N as N → ∞

- When we talk of a 'stationary time series process' we shall mean a weakly stationary purely indeterministic process.

- Example
  $X_t = Y_{t-1} + Y_t$ is a Stationary time series process

1.4 White Noise Process

White Noise Process e_t is a series of uncorrelated random variables.

For time series we assume the mean of White Noise Process to be zero, $\mathrm{E}(e_t) = 0$

$$\gamma_k = \operatorname{cov}(e_t, e_{t+k}) \quad \begin{aligned} &= 0;\ K > 0 \\ &= \sigma^2;\ K = 0 \end{aligned}$$

Sequence of normal random variable are an important representative of WNP.

A white noise process with mean zero is used to model error in Time Series.

Question

Let Y_t be a sequence of independent standard normal random variables. Determine if the following process is stationary time series (given the definition above).

$$X_t = X_{t-1} + Y_t$$

Solution

Here we have:

$$E(X_t) = E(X_{t-1} + Y_t) = E(X_{t-1}) + E(Y_t) = E(X_{t-1})$$

So the process has a constant mean. However:

$$\text{var}(X_t) = \text{var}(X_{t-1} + Y_t) = \text{var}(X_{t-1}) + \text{var}(Y_t) = \text{var}(X_{t-1}) + 1$$

Here we are using the fact that Y_t is a sequence of independent standard normal random variables. Since the variance is not constant, the process is not stationary.

1.5 Auto Covariance and Auto Correlation

Auto Covariance

If Time Series is stationary, covariance depends only on the lag k that is

$$\gamma_k = \text{cov}(X_t, X_{t+k})$$

Depends only on time difference & not specific points in time

Auto Correlation Function

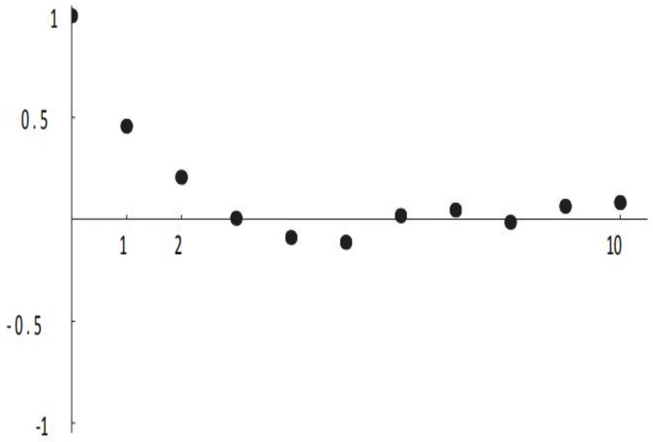
$$\rho_k = \frac{\text{cov}(X_t, X_{t+k})}{\sqrt{(V(X_t) * V(X_{t+k})}} = \frac{\gamma_k}{\gamma_o}$$

For Purely Indeterministic Process as $k \to \infty$; $\rho_k \to 0$

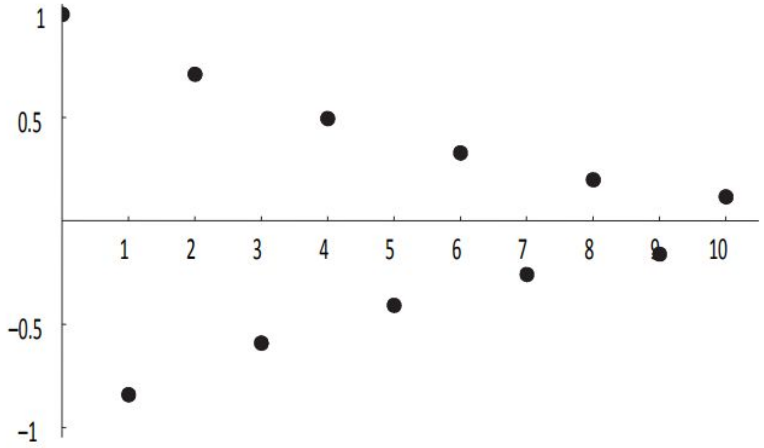
1.5 Auto Covariance and Auto Correlation

Correlograms – Autocorrelation functions are the most commonly used statistic in time series analysis. A lot of information about a time series can be deduced from a plot of the sample autocorrelation function (as a function of the lag). Such a plot is called a correlogram.

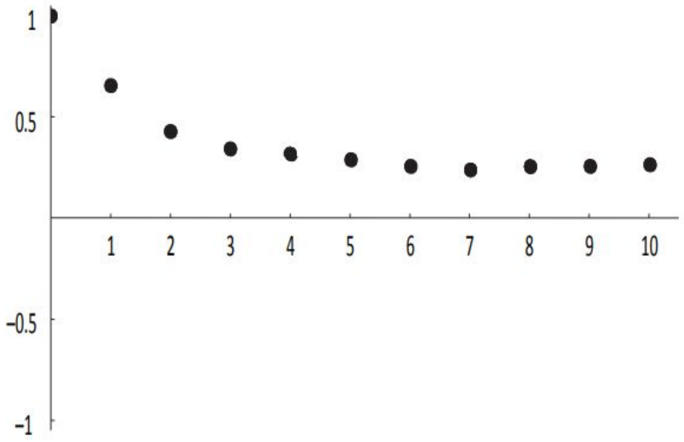
Stationary series



Alternating series



Series with trend



1.6 Partial Auto Correlation Function (PACF)

Conditional auto correlation of X_{t+k} with X_t given $X_{t+1}, X_{t+2}, X_{t+3}$ is ϕ_k

$$Corr(X_t, X_{t+k} \mid X_{t+1}, X_{t+2}, \ldots \ldots X_{t+k-1})$$

$$\phi_1 = \rho_1$$

$$\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$$

These formulae can be found on page 40 of the Tables.

2 Linear Models of Time Series

The main linear models used for modelling stationary time series are:

- **Autoregressive process (AR)**
- **Moving average process (MA)**
- **Autoregressive moving average process (ARMA).**

The definitions of each of these processes, presented below, involve the standard zero mean white noise process

2.1 Auto Regressive process - AR(p)

A Time Series process X is an $AR(p)$ process if it depends on past p terms of series

$$\boldsymbol{X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots \ldots \alpha_p\left(X_{t-p} - \mu\right) + e_t}$$

This is an AR(p) process with mean μ

Thus, the autoregressive model attempts to explain the current value of X as a linear combination of past values with some additional externally generated random variation.

The similarity to the procedure of linear regression is clear and explains the origin of the name 'autoregression'.

2.1 AR(1) Process

AR(1) process

$$X_t = \mu + \alpha(X_{t-1} - \mu) + e_t$$

where $e_t \sim WNP$

Mean of AR(1) process.

$$E(X_t) = \mu + \alpha^t(\mu_o - \mu)$$

Variance of AR(1) process

$$\mathrm{V}(X_t) = \alpha^2\left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right) + \alpha^{2t}\mathrm{Var}(X_o)$$

where, as before, σ^2 denotes the common variance of the white noise terms $\{e_t\}$.

Auto Covariance Function

$$\gamma_k = \alpha^k \gamma_0$$

2.1 The backwards shift operator, B, and the difference operator, ∇

The **backwards shift operator, B** , acts on the process X to give a process BX such that:

$$\mathbf{B}\boldsymbol{X_t} = \boldsymbol{X_{t-1}}$$

Operators can be applied repeatedly

$$B^2X_t = X_{t-2} \quad and \quad B^rX_t = X_{t-r}$$

The **difference operator, ∇**, is defined as $\boldsymbol{\nabla = 1 - B}$

$$\boldsymbol{\nabla X_t = X_t - X_{t-1}}$$

Operators can be applied repeatedly

$$\begin{aligned}\nabla^2 X_t &= \nabla(\nabla X_t - \nabla X_{t-1}) \\ &= X_t - X_{t-1} - X_{t-1} + X_{t-2} \\ &= X_t - 2X_{t-1} + X_{t-2}\end{aligned}$$

The usefulness of both operators will become apparent in later sections.

2.1 AR(p) Process

The equation of the more general $AR(p)$ process is:

$$\boldsymbol{X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \cdots + \alpha_p\left(X_{t-p} - \mu\right) + e_t}$$

or, in terms of the backwards shift operator:

$$\boldsymbol{\left(1 - \alpha_1 B - \alpha_2 B^2 - \cdots - \alpha_p B^p\right)(X_t - \mu) = e_t}$$

2.1 AR(p) Process

Condition for stationarity of an AR(p) process

If the time series process X given by AR(p) is stationary, then the roots of the equation:

$$1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p = 0$$

are **all greater than 1 in absolute value**.

(The polynomial $1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p$ is called the characteristic polynomial of the autoregression.)

2.1 Yule Walker Equations

Often exact values for the γ_k are required, entailing finding the values of the constants A_k. we have:

$$\operatorname{cov}(X_t, X_{t-k}) = \alpha_1 \operatorname{cov}(X_{t-1}, X_{t-k}) + \cdots + \alpha_p \operatorname{cov}\left(X_{t-p}, X_{t-k}\right) + \operatorname{cov}(e_t, X_{t-k})$$

which can be re-expressed as:

$$\boldsymbol{\gamma_k = \alpha_1 \gamma_{k-1} + \alpha_2 \gamma_{k-2} + \cdots + \alpha_p \gamma_{k-p} + \sigma^2 1_{\{k=0\}}}$$

for $0 \leq k \leq p$.

These are known as the Yule-Walker equations. Here the notation $1_{\{k=0\}}$ denotes an indicator function, taking the value 1 if $k = 0$, the value 0 otherwise.

We can make equations for $\gamma_{0,}\gamma_1, \gamma_{2,}\ \gamma_3$ so on, and solve them simultaneously to find their values.

2.1 Behavior of the PACF for an AR(p) process

For an $AR(p)$ process:

$$\boldsymbol{\phi_k = 0 \text{ for } k > p}$$

This property of the PACF is characteristic of autoregressive processes and forms the basis of the most frequently used test for determining whether an AR(p) model fits the data.

It would be difficult to base a test on the ACF as the ACF of an autoregressive process is a sum of geometrically decreasing components.

2.2 The first-order moving average model, MA(1)

A first-order moving average process, denoted $MA(1)$, is a process given by:

$$\boldsymbol{X_t = \mu + e_t + \beta e_{t-1}}$$

The mean of this process is $\mu_t = \mu$.

The variance and autocovariance are:

$$\gamma_0 = \text{var}(e_t + \beta e_{t-1}) = (1 + \beta^2)\sigma^2$$
$$\gamma_1 = \text{cov}(e_t + \beta e_{t-1}, e_{t-1} + \beta e_{t-2}) = \beta\sigma^2$$
$$\gamma_k = 0 \text{ for } k > 1$$

Hence the ACF of the $MA(1)$ process is:

$$\rho_0 = 1$$
$$\rho_1 = \frac{\beta}{1 + \beta^2}$$
$$\rho_k = 0 \text{ for } k > 1$$

2.2 Stationarity & Invertibility

An $MA(1)$ process is stationary regardless of the values of its parameters.
The parameters are nevertheless usually constrained by imposing the condition of invertibility.

The defining equation of the $MA(1)$ may be written in terms of the backwards shift operator:

$$\boldsymbol{X - \mu = (1 + \beta B)e}$$

In many circumstances an autoregressive model is more convenient than a moving average model.

We may rewrite MA(1) as:

$$(1 + \beta B)^{-1}(X - \mu) = e$$

and use the standard expansion of $(1 + \beta B)^{-1} = 1 - \beta B + \beta^2 B^2 - \beta^3 B^3 + \cdots$ to give:

$$\boldsymbol{X_t - \mu - \beta(X_{t-1} - \mu) + \beta^2(X_{t-2} - \mu) - \beta^3(X_{t-3} - \mu) + \cdots = e_t}$$

MA(1) -> AR(∞)
The expansion referred to here is given on page 2 of the Tables.

2.2 Condition for Invertibility

The original moving average model has therefore been transformed into an autoregression of infinite order. But this procedure is only valid if the sum on the left-hand side is convergent, in other words if $|\beta| < 1$.

When this condition is satisfied the MA(1) is called invertible.

2.2 MA(q) Process

The defining equation of the general q th order moving average is, in backwards shift notation:

$$X - \mu = (1 + \beta_1 B + \beta_2 B^2 + \cdots + \beta_q B^q)e$$

In other words, it is:

$$\boldsymbol{X_t - \mu = e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}}$$

Moving average processes are always stationary, as they are a linear combination of white noise, which is itself stationary.

2.2 Condition for Invertibility

The process X defined by the equation:

$$x_t - \mu = e_t + \beta_1 e_{t-1} + \beta_2 e_{t-2} + \cdots + \beta_q e_{t-q}$$

is invertible if and only if the roots of the equation:

$$1 + \beta_1 z + \beta_2 z^2 + \cdots + \beta_q z^q = 0$$

are **all strictly greater than 1 in magnitude.**

This is equivalent to saying that the value e_t can be written explicitly as a (convergent) sum of X values.

Question

CS2A September 2022 Q2

Consider the time series process, Xt, given by:

$$X_t = aX_{t-1} + \frac{1}{2}X_{t-2} + e_t + be_{t-1}$$

where e_t is a sequence of independent and identically distributed $N(0, \sigma^2)$ random variables.

Determine the values of the parameters a and b such that Xt is:

(i) stationary. [5]

(ii) invertible. [2]

Solution

(i)
X_t is stationary if and only if the roots of the characteristic polynomial

$$1 - a * \lambda - \tfrac{1}{2} * \lambda^2$$

are both greater than 1 in magnitude
For $\lambda = 1$ to be a root, $a = \frac{1}{2}$
For $\lambda = -1$ to be a root, $a = -\frac{1}{2}$
If $a = 0$, then the characteristic polynomial reduces to $1 - \frac{1}{2} * \lambda^2$
This has roots $\lambda = \sqrt{2}$ and $-\sqrt{2}$, which are greater than 1 in magnitude
Stationarity therefore holds for $a = 0$
Overall, stationarity holds if and only if $|a| < \frac{1}{2}$

Solution

(ii)
X_t is invertible if and only if the value of *lambda* satisfying
1 + *b* * *lambda* = 0
is greater than 1 in magnitude,
i.e. if and only if -1 / *b* is greater than 1 in magnitude
Hence invertibility holds if and only if abs(*b*) < 1

Question

CT6 September 2017 Q10

Let $X_t = a + bt + Y_t$, where Y_t is a stationary time series, and a and b are fixed non-zero constants.

(i) Show that X_t is not stationary. [2]

Let $\Delta X_t = X_t - X_{t-1}$.

(ii) Show that ΔX_t is stationary. [1]

(iii) Determine the autocovariance values of ΔX_t in terms of those of Y_t. [4]

Now assume that Y_t is an MA(1) process, i.e. $Y_t = \varepsilon_t + \beta\varepsilon_{t-1}$

(iv) Set out an equation for ΔX_t in terms of b, β, ε_t and L, the lag operator. [1]

(v) Show that ΔX_t has a variance larger than that of Y_t. [4]

[Total 12]

Solution

i) Since $E(Y_t) = \mu_Y$ is constant for each t, $E(X_t) = a + bt + \mu_Y$.

Since this mean depends on t then the time series X_t is not stationary.

(ii) $\Delta X_t = b + \Delta Y_t$ (and since Y_t is stationary ΔX_t is)

Solution

(iii) And the covariance function is

$$\mathrm{Cov}(\Delta X_t, \Delta X_{t-s}) = \mathrm{Cov}(Y_t - Y_{t-1}, Y_{t-s} - Y_{t-s-1})$$

$$= \mathrm{Cov}(Y_t, Y_{t-s}) + \mathrm{Cov}(Y_{t-1}, Y_{t-1-s}) - \mathrm{Cov}(Y_t, Y_{t-s-1}) - \mathrm{Cov}(Y_{t-1}, Y_{t-s})$$

$$= \gamma^Y(s) + \gamma^Y(s) - \gamma^Y(s+1) - \gamma^Y(s-1)$$

$$= 2\gamma^Y(s) - \gamma^Y(s+1) - \gamma^Y(s-1)$$

Where $\gamma^Y(s)$ represents the autocovariance of Y at s.

Solution

(iv) $Y_t = \varepsilon_t + \beta\varepsilon_{t-1}$, where ε_t is white noise with variance σ^2 then

$\Delta X_t = b + \varepsilon_t + \beta\varepsilon_{t-1} - \varepsilon_{t-1} - \beta\varepsilon_{t-2}$ [1]

or

$\Delta X_t = b + (1 - L)(1 + \beta L)\,\varepsilon_t$. [1]

[Max 1]

(v) In particular $Y_t = \varepsilon_t + \beta\varepsilon_{t-1}$ the corresponding auto-covariance function is

$\gamma^Y(0) = (1+\beta^2)\sigma^2$ and $\gamma^Y(1) = \beta\sigma^2$. [2]

So from (iii) $var(\Delta(X)) = 2\gamma^Y(0) - 2\gamma^Y(1) = 2(1+\beta^2)\sigma^2 - 2\beta\sigma^2 =$
$(1+\beta^2)\sigma^2 + (1-\beta)^2\sigma^2 > (1+\beta^2)\sigma^2 = \gamma^Y(0)$ [2]

2.3 The autoregressive moving average process

A combination of the moving average and autoregressive models, **an ARMA model** includes direct dependence of X_t on both past values of X and present and past values of e.

The defining equation is:

$$\boldsymbol{X_t = \mu + \alpha_1(X_{t-1} - \mu) + \cdots + \alpha_p\left(X_{t-p} - \mu\right) + e_t + \beta_1 e_{t-1} + \cdots + \beta_q e_{t-q}}$$

or, in backwards shift operator notation:

$$\left(1 - \alpha_1 B - \cdots - \alpha_p B^p\right)(X - \mu) = \left(1 + \beta_1 B + \cdots + \beta_q B^q\right)e$$

Autoregressive and moving average processes are special cases of ARMA processes. AR(p) is the same as ARMA (p, 0). MA(q) is the same as ARMA (0, q).

2.3 Stationarity & Invertibility for ARMA(p, q)

- To check the stationarity of an ARMA process, we just need to examine the autoregressive part.
- The moving average part (which involves the white noise terms) is always stationary. The test is the same as for an autoregressive process – we need to determine the roots of the characteristic equation formed by the X terms.
  The process is stationary if and only if all the roots are strictly greater than 1 in magnitude.
- Similarly, we can check for invertibility by examining the roots of the characteristic equation that is obtained from the white noise terms.
  The process is invertible if and only if all the roots are strictly greater than 1 in magnitude.

2.4 ARIMA(p, d, q)

Definition of an ARIMA process
If X needs to be differenced at least $\boldsymbol{d}$ times in order to reduce it to stationarity and if the dth difference $Y = \nabla^d X$ is an $\mathrm{ARMA}(p, q)$ process, then X is termed an $\mathrm{ARIMA}(p, d, q)$ process.

Equation:

In terms of the backwards shift operator, the equation of the $\mathrm{ARIMA}(p, d, q)$ process is:

$$\left(\mathbf{1} - \boldsymbol{\alpha}_{\mathbf{1}}\boldsymbol{B} - \cdots - \boldsymbol{\alpha}_{\boldsymbol{p}}\boldsymbol{B}^{\boldsymbol{p}}\right)(\mathbf{1} - \boldsymbol{B})^{\boldsymbol{d}}(\boldsymbol{X} - \boldsymbol{\mu}) = \left(\mathbf{1} + \boldsymbol{\beta}_{\mathbf{1}}\boldsymbol{B} + \cdots + \boldsymbol{\beta}_{\boldsymbol{q}}\boldsymbol{B}^{\boldsymbol{q}}\right)\boldsymbol{e}$$

An $\mathrm{ARIMA}(p, d, q)$ process is $I(d)$.

We can think of the classification $\mathrm{ARIMA}(p, d, q)$ as:

$$AR(p)I(d)MA(q)$$

3 Markov Process

- If future development is determined based on present value inly then process is said to have Markov property and hence the process could be called a Markov process
- Thus, Markov processes are the natural stochastic analogs of the deterministic processes described by differential and difference equations.

- AR(1) is a Markov process
- AR(p) is a not a Markov Process
- MA(1) is not a Markov process [since MA(1) $\rightarrow$ AR(∞)]

Question

CS2A September 2022 Q3

Consider the following stochastic process:

$$X_t = \sum_{i=0}^{\infty} 0.5^i e_{t-i}$$

where e_j is a sequence of independent and identically distributed random variables with a mean of zero and variance σ^2, for $j = 0, \pm 1, \pm 2, \ldots$

(i) Determine whether X_t is stationary and satisfies the Markov property. [4]

(ii) Determine whether your conclusions from part (i) also apply to the process $Y_t = X_t - 0.3X_{t-1}$. [6]

[Total 10]

Solution

(i)

X_t is stationary since it is an AR(1) process. In particular

$E(X_t)=E\sum_0^{\infty} 0.5^i E(e_{t-i})=0$ [½]

$Cov(X_t,X_{t-s})= E(\sum_0^{\infty} 0.5^i E(e_{t-i}),$

$\sum_0^{\infty} 0.5^i E(e_{t-s-i}))=0+ 0.5^s E(\sum_0^{\infty} 0.5^i 0.5^i e^2_{t-s-i})$

$=0.5^s \sigma^2 /(1-0.5^2)$ [1½]

As $X_t=0.5 X_{t-1}+e_t$, the distribution of X_t depends on X_{t-1} and only the information at time t-1 and not any other r.v. before that point [1]

Hence we have the Markov property [1]

Solution

(ii)
The stationarity of Yt is implied from that of Xt as
E(Y_t)= E(X_t)-0.3 E(X_{t-1})=0

and

Cov(Y_t,Y_{t-s})= Cov(X_t-0.3 X_{t-1},X_{t-s} -0.3 X_{t-s-1})=

Cov(Xt,X_{t-s})-0.3 Cov(Xt,X_{t-s-1}) -0.3 Cov(X{t-1},X_{t-s})+0.09

Cov(X{t-1},X_{t-s-1})

All these four components do not depend on t due to the stationarity of Xt

For the Markov property Yt however this is not the case:
As Xt=0.5 X_{t-1}+e_t,
Y_t=0.5 X_t-1+et-0.3 X_{t-1}=et+0.2 X_{t-1}= et+0.2 *0.5 X_{t-2}+0.2 e_{t-1}

Substituting X_{t-2}=1/0.2*(Y_{t-1}-e_{t-1})
Y_t=0.5 Y_{t-1}+e_t-0.3 e_{t-1}

In this form one can see that the prediction for Y_t depends not only on Y_{t-1} but also on the information contained in e_{t-1}
Hence the Markov property is NOT satisfied