Lecture 1

INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: TY BSc

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Chapter: Unit 3 Chapter 1

Chapter Name: Time Series Models Analysis and Forecasting-

Today’s Agenda

1. Introduction to Stochastic Processes and its Main Characteristics
    1. Univariate Time Series Processes
    2. Stationarity
    3. Purely Indeterministic Processes
    4. White Noise Process
    5. Auto Covariance and Auto Correlation
2. Time Series Processes
    1. Auto Regressive (AR) Process
    2. Moving Average Process
    3. ARMA Process
    4. ARIMA Process
3. Markov Process

Introduction

Time Series is a Stochastic Process indexed in

- Discrete Time Domain
- Continuous State Space

Time series data is a collection of observations obtained through repeated measurements over time

Example

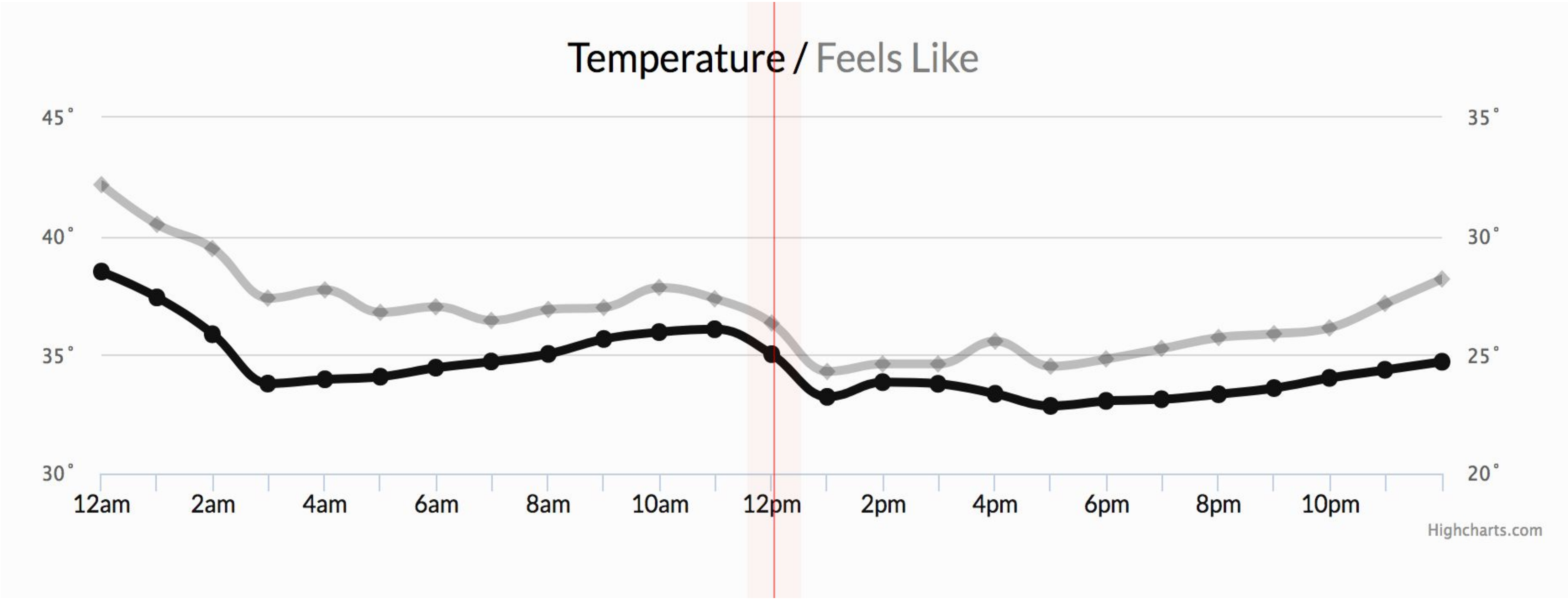
- Closing Price of a Share
- Inflation Rate every quarter
- Temperature on a given day

Introduction Contd

Why do we need to model Time Series ?

- To help understand the process better
- To be able to forecast / predict future behaviour
- To improve decision making

Plot of a Time Series Process



1.1 Univariate Time Series Process

- Observe a single process at a sequence of different times
- Continuous State Space
- Discrete Time Set

- The term "univariate time series" refers to a time series that consists of **single (scalar) observations recorded sequentially over equal time increments**. ... If the data are equi-spaced, the time variable, or index, does not need to be explicitly given.

1.2 Stationarity

- Stationarity means that statistical properties of the stochastic process remain unchanged.
- Stationarities can be of two types
  Strong / Strictly Stationary
  Weak Stationary

A process {Xt} is
Strict Stationarity :- If the joint Distribution of $(X_{t1}, X_{t2}, X_{t3} \ldots X_{tn})$ &$(X_{t1+k}, X_{t2+k} \ldots X_{tn+k})$ **is same i. e. all statistical properties remain the same.**

Weak Stationarity if,
- Mean i.e. E(X_t) is constant
- Covariance depends only on the time lag
- As covariance depends only on the time lag it implies that the variance of the process V(X_t) (Covariance with lag 0) is constant

1.3 Purely Indeterministic Process

The process {Xt} is a purely indeterministic process if knowledge of X_1 ,X_2,X_3 is progressively less useful at predicting the value of X_N as N → ∞

- When we talk of a 'stationary time series process' we shall mean a weakly stationary purely indeterministic process.

- Example
  $X_t = Y_{t-1} + Y_t$ is a Stationary time series process

1.4 White Noise Process

- White Noise Process $\boldsymbol{e_t}$ is a series of uncorrelated random variables.
- For time series we assume the mean of White Noise Process to be zero

- E(e_t) = 0

- $$\gamma_k = cov(e_t, e_{t+k}) = 0 \quad ; \; K = !0$$
  $$= \sigma^2 ; \; K = 0$$

- Sequence of normal random variable are an important representative of WNP
- A white noise process with mean zero is used to model error in Time Series

Question

Let Y_t be a sequence of independent standard normal random variables. Determine which of the following processes are stationary time series (given the definition above).

(i) $X_t = \sin(\omega t + U)$, where U is uniformly distributed on the interval $[0, 2\pi]$

(ii) $X_t = \sin(\omega t + Y_t)$

(iii) $X_t = X_{t-1} + Y_t$

1.5 Auto Covariance and Auto Correlation

Auto Covariance

- If Time Series is stationary, covariance depends only on the lag k that is $\gamma_k = cov(X_t, X_{t+k})$
- Depends only on time difference & not specific points in time

- Auto Correlation Function

$$\rho_k = \frac{cov(X_t, X_{t+k})}{\sqrt{(V(X_t) * V(X_{t+k})}} = \frac{\gamma_k}{\gamma_o}$$

- For Purely Indeterministic Process as k → ∞ ; $\rho_k \rightarrow 0$

1.5 Auto Covariance and Auto Correlation

Correlograms

Stationary series

Alternating series

Series with trend

1.5 Auto Covariance and Auto Correlation

Question - Correlograms

Plot correlogram of average daytime temperature in successive months in Mumbai

Partial Auto Correlation Function PACF

Conditional auto correlation of X_{t+k} with X_t given $X_{t+1}, X_{t+2}, X_{t+3}$ is ϕ_k

- Corr($X_t, X_{t+k} \mid X_{t+1}, X_{t+2}, \ldots \ldots X_{t+k-1}$)
- $\phi_1 = \rho_1$
- $\phi_2 = \frac{\rho_2 - \rho_1^2}{1 - \rho_1^2}$

Backward Shift Operator & Difference Operator

- Backward Shift Operator

  $B X_t = X_{t-1}$

  $B^2 X_t = X_{t-2}$

  $B^r X_t = X_{t-r}$

For constant terms μ , $B\mu$, $B^2\ \mu = \mu$,$B^r\ \mu = \mu$

- Difference Operator

  $\nabla X_t = X_t - X_{t-1}$

  $$\nabla^2 X_t = \nabla(\nabla X_t - \nabla X_{t-1})$$
  $$= X_t - X_{t-1} - X_{t-1} + X_{t-2}$$
  $$= X_t - 2X_{t-1} + X_{t-2}$$

  $\nabla = 1 - B$

2.1 Auto Regressive Model & AR(p) process

- A Time Series process X is an AR(p) process if it depends on past p terms of series
- $X_t = \mu + \alpha_1(X_{t-1} - \mu) + \alpha_2(X_{t-2} - \mu) + \ \ldots \ldots \alpha_p\left(X_{t-p} - \mu\right) + e_t$
  in an AR(p) process with mean μ

Xt = $\alpha_1 X_{t-1} + \alpha_2 X_{t-2} + \ \ldots \ldots \alpha_p X_{t-p} + e_t$ *is an* AR(p) process with mean zero.

AR(1) Process

- AR(1) process X_t
- $X_t = \mu + \alpha(X_{t-1} - \mu) + e_t$
  where $e_t \sim WNP$

- Mean of AR(1) process.
- $E(X_t) = \mu + \alpha^t(\mu_o - \mu)$

- Variance of AR(1) process
- $V(X_t) = \alpha^2 \left(\frac{1-\alpha^{2t}}{1-\alpha^2}\right) + \alpha^{2t} Var(X_o)$

- Auto Covariance Function
- $\gamma_k = \alpha^k \gamma_0$

Condition for Stationarity

AR(p) Process

- Characteristic equation

- Condition for Stationarity

Determine whether the process $X_n = X_{n-1} - \frac{1}{2}X_{n-2} + e_n$ is stationary.

Yule Walker Equation

$$\gamma_k = \alpha_1 * \gamma_{k-1} + \alpha_2 \gamma_{k-1} + \cdots \ldots \alpha_p * \gamma_{k-p} + \sigma^2 \;\; if\,(K = 0)$$

Question

Consider the time series model

$$(1-\alpha B)^3 X_t = e_t$$

where B is the backwards shift operator and e_t is a white noise process with variance σ^2.

(i) Determine for which values of α the process is stationary. [2]

Now assume that $\alpha = 0.4$.

(ii) (a) Write down the Yule-Walker equations.

(b) Calculate the first two values of the auto-correlation function ρ_1 and ρ_2. [7]

(iii) Describe the behaviour of ρ_k and the partial autocorrelation function ϕ_k as $k \to \infty$. [3]

HW

Consider the following time series model:

$$Y_t = 1 + 0.6Y_{t-1} + 0.16Y_{t-2} + \varepsilon_t$$

where ε_t is a white noise process with variance σ^2.

(i) Determine whether Y_t is stationary and identify it as an ARMA(p,q) process. [3]

(ii) Calculate $E(Y_t)$. [2]

(iii) Calculate for the first four lags:

- the autocorrelation values $\rho_1, \rho_2, \rho_3, \rho_4$ and
- the partial autocorrelation values $\psi_1, \psi_2, \psi_3, \psi_4$. [7]

Auxiliary equation and difference equation

For $ay_t = by_{t-1} + cy_{t-2}$

Ex. – $X_t = 5/6\ X_{t-1} - 1/6\ X_{t-2} + e_t$

Q. – Give the general form of ACF of

$$X_n = 0.8\ X_{n-1} - 0.1\ X_{n-2} + e_n$$

2.2 Moving Average Process

MA(q) Process

- A time series process X is a MA(q) process if it can be written as weighted average of the past ‘q’ white noise terms (i.e error terms)

- $X_t = \mu + e_t + \beta_1 * e_{t-1} + \beta_2 e_{t-2} + \cdots \beta_q * e_{t-q}$

  mean (μ)

- $X_t = e_t + \beta_1 * e_{t-1} + \beta_2 e_{t-2} + \cdots \beta_q * e_{t-q}$

  mean 0

1.2 MA(1) process

- MA(1) process
- $X_t = \mu + e_t + \beta_1 * e_{t-1}$
- Mean E(X_t) = μ
- Variance V(X_t) =
- Auto Correlation Function

1.2 MA(q) process

Assuming $e_n \sim N(0,1)$, calculate the autocovariance and autocorrelations of the following process and mention the type of process:

$$X_n = 3 + e_n - e_{n-1} + 0.25\, e_{n-2}$$

HW

$$X_n = 1 + e_n - 5\, e_{n-1} + 6\, e_{t-2}$$

Invertibility & Stationarity

- A time series is invertible if the white noise terms e_t is a convergent sum of the X terms
- MA(1) → AR(∞)
- Invertibility is a desirable characteristic as it enables us to calculate the residual / error term & hence , analyse goodness of fit of the model
- Similarly, a Time Series X is stationary if X term is a convergent sum of the e_t terms
- AR(1) → MA(∞)

Question

Determine whether the process $X_t = 2 + e_t - 5e_{t-1} + 6e_{t-2}$ is invertible.

2.3 ARMA (p,q) process

- A Time Series X is ARMA (p,q) if it is the sum of AR(p) process & MA(q) process
- Xt = $\mu + \alpha_1(X_{t-1} - \mu) + \ldots \ldots \alpha_p \left(X_{t-p} - \mu\right) + e_t + \beta_1 e_{t-1} + \ldots + \beta_q e_{t-q}$

Conditions for Stationarity and Invertibility

Show that the process $12X_t = 10X_{t-1} - 2X_{t-2} + 12e_t - 11e_{t-1} + 2e_{t-2}$ is both stationary and invertible.

2.3 ARMA (1,1) process

Xt = $\alpha_1(X_{t-1} - \mu) + e_t + \beta_1 e_{t-1}$

Find autocovariance and autocorrelation function of the process

2.4

ARIMA (p,d,q) process → Auto Regressive Integrated Moving Average

- A Time Series is an ARIMA(p ,d ,q) process if it needs to be differenced 'd' times to reduce it to a stationary process.
- i.e. if Y = $\nabla^d X$ is an ARMA(p ,q) process, then X is an ARIMA(p ,d ,q) process

2.4 ARIMA (p,d,q) process → Auto Regressive Integrated Moving Average

$X_t = 0.6\, X_{t-1} + 0.3\, X_{t-2} + 0.1\, X_{t-3} + e_t - 0.25\, e_{t-1}$

Check whether it is an ARIMA process, and if yes, solve for p,d,q

HW

$2\, X_t = 7\, X_{t-1} - 9\, X_{t-2} + 5\, X_{t-3} - X_{t-4} + e_t - e_{t-2}$

Question

(i) Show that the relationship $Y_t = 0.7Y_{t-1} + 0.3Y_{t-2} + Z_t + 0.7Z_{t-1}$ (where the Z's denote white noise) defines an $ARIMA(1,1,1)$ process.

(ii) Show carefully that the relationship $S_t = 1.5S_{t-1} + 0.5S_{t-3} + Z_t + 0.5Z_{t-1}$ *cannot* be expressed as an $ARIMA(1,2,1)$ process.

3.1 Markov Process

- If future development is determined based on present value inly then process is said to have Markov property and hence the process could be called a Markov process
- Thus, Markov processes are the natural stochastic analogs of the deterministic processes described by differential and difference equations.

- AR(1) is a Markov process
- AR(p) is a not a Markov Process
- MA(1) is not a Markov process [since MA(1) $\rightarrow$ AR(∞)]

Question

Let $X_n = e_n + e_{n-2}$ be an MA(2) process where $e_n \sim N(0,1)$.

(i) Calculate $P\left(X_n \geq 0 \middle| X_{n-1} \leq 0\right)$.

(ii) Compare your answer to (i) with $P\left(X_n \geq 0 \middle| X_{n-1} \leq 0, X_{n-2} \leq 0\right)$ and hence comment on whether the process is Markov.

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