INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Class: MSc

Subject : Statistical and Risk Modelling - 4

Chapter: Unit 2

Chapter Name: Extreme Value Theory

Today’s Agenda

1. Introduction
   1. Introduction to Extreme events
   2. Heteroscedasticity

2. Extreme Value Theory
   1. Generalised extreme value (GEV) distribution

3. Peak over threshold exceedances

4. Measures of Tail Weights

1.1 Introduction to Extreme Events

Extreme events
Events that have a low probability of occurring and a high financial impact i.e. low frequency and high severity.

Example :-

- A loss due to natural calamity
- A stock market crash.

Modelling such extreme events proposes a set of difficulties.
What do you think can be the difficulties encountered while modelling extreme events?

1.1 Introduction to Extreme Events

Difficulties in modelling extreme events

- No data / very less data for extreme events.
- Non normality of financial returns or losses as these event distributions have fatter tails and sharper peaks
- Probability of extreme events is underestimated as normal distribution has narrow tails.

1.2 Heteroscedasticity

Let’s revise Kurtosis

Many types of financial data tend to be narrowly peaked in the centre of the distribution and to have fatter tails than the normal distribution. This shape of distribution is known as leptokurtic.

For example, when share prices are modelled, large price movements occur more frequently than predicted by the normal distribution. So the normal distribution may be unsuitable for modelling the large movements in the tails.

The word ‘leptokurtic’ is a measure of the kurtosis of a distribution, which is the fourth standardised central moment of a distribution:

- $K = 3$ is mesokurtic distribution [Normal Distribution]
- $K > 3$ is leptokurtic distribution [Sharper Peak]
- $K < 3$ is platykurtic distribution [Flatter Peak]

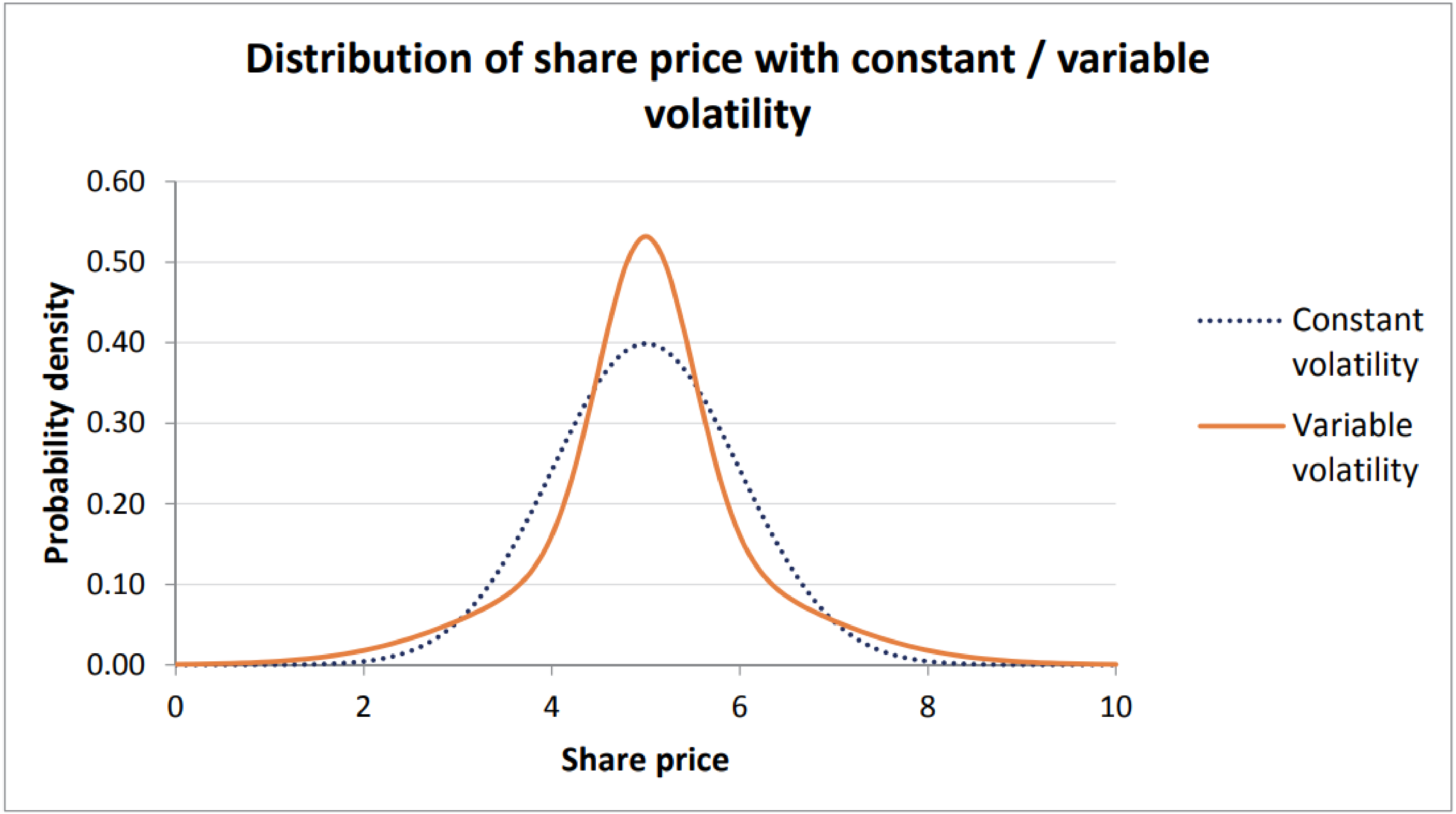
1.2 Heteroscedasticity

Asset Return Volatility is not a constant (clustering volatility) but it changes stochastically with time. This property is known as heteroscedasticity.

The graph below compares two distributions for the price of a share in one year's time:

- a $N(5,\sigma^2)$ distribution with constant volatility, $\sigma = 1$
- a $N(5,\sigma^2)$ distribution where the volatility is heteroscedastic, *ie* $\sigma = 0.5$ and $\sigma = 1.5$ with equal probability.

1.2 Heteroscedasticity



2 Extreme value theory

Fortunately, better modelling of the tails of the data can be done through the application of extreme value theory. The key idea of extreme value theory is that the asymptotic behaviour of the tails of most distributions can be accurately described by certain families of distributions.

More specifically, the maximum values of a distribution (when appropriately standardised) and the values exceeding a specified threshold (called threshold exceedances) converge to two families of distributions as the sample size increases.

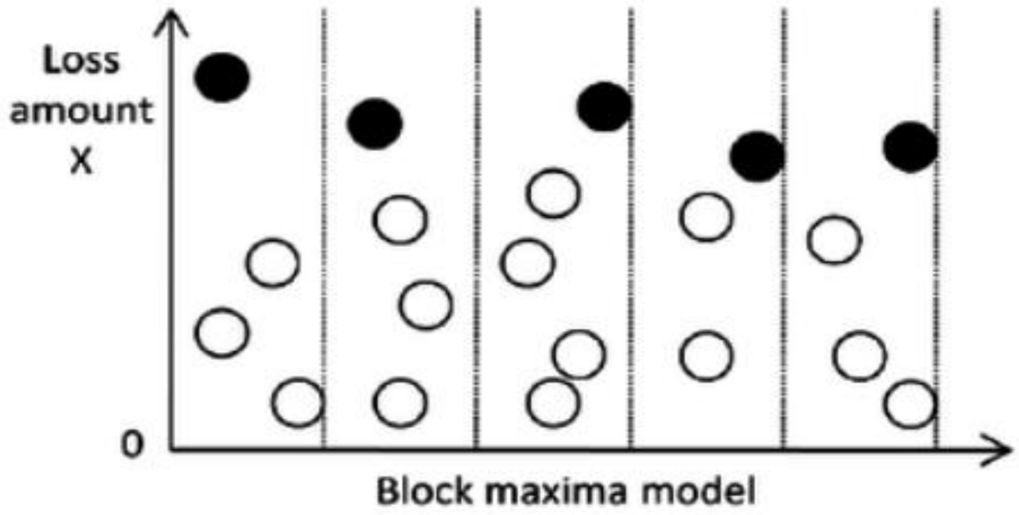
These two families of distributions are:
- generalised extreme value distributions, and
- generalised Pareto distributions.

2.1 Generalised Extreme value (GEV) distribution

Block Maxima

One approach is to look at $X_M = \max\{X_1, X_2, \dots, X_n\}$, the maximum value in a set of n values. This is referred to as a block maximum.

X_m = max value of X in a block
m = no of blocks
n = no of values in a block
$X_m = \max(X_1, X_2, \dots X_n)$
X_m is the block maxima



2.1 Generalised Extreme value (GEV) distribution

The dataset below shows the claim amounts in £000s in respect of a commercial property portfolio over a period of a year.

Claim number	Claim amount	Claim number	Claim amount	Claim number	Claim amount	Claim number	Claim amount
1	9	17	12	33	19	49	118
2	28	18	35	34	17	50	55
3	20	19	12	35	66	51	14
4	8	20	75	36	55	52	94
5	102	21	80	37	81	53	54
6	152	22	42	38	140	54	81
7	23	23	9	39	64	55	62
8	108	24	122	40	9	56	83
9	42	25	145	41	9	57	23
10	12	26	13	42	36	58	19
11	110	27	16	43	185	59	55
12	9	28	113	44	135	60	104
13	22	29	9	45	25		
14	37	30	8	46	16		
15	147	31	12	47	55		
16	128	32	84	48	31		

2.1 Generalised Extreme value (GEV) distribution

Question

i) Determine the values of X_M where the block size is:
(a) n = 5
(b) n = 10

(ii) Comment on the trade-off between the block size and the values of X_M that will be used to fit the extreme value distribution.

2.1 Generalised Extreme value (GEV) distribution

Solution

(i)(a) The values of X_M are $\{102, 152, 147, 128, 145, 113, 84, 140, 185, 118, 94, 104\}$.

(i)(b) The values of X_M are $\{152, 147, 145, 140, 185, 104\}$.

(ii) The larger the block size, the fewer the number of blocks (*eg* when $n = 10$ there are six blocks whereas when $n = 5$ there are twelve blocks). The fewer the number of blocks, the fewer (but more 'extreme') the values of X_M that will be used to fit the extreme value distribution.

2.1 Generalised Extreme value (GEV) distribution

Block Maxima

If we look at a number of such blocks, we find that these maximum values can be standardised in a similar way, ie we can calculate expressions of the form $\frac{X_M - \alpha_n}{\beta_n}$ that can be approximated by a particular type of distribution - called an extreme value distribution.

Distribution of the (standardised) maximum values

$\mathrm{F}(x)$, the cumulative distribution function of the block maximum is:

$$\begin{aligned} P(x_M \leq x) \quad &= P(x_1 \leq x, x_2 \leq x, \dots, x_n \leq x) \\ &= P(x_1 \leq x) P(x_2 \leq x) \dots P(x_n \leq x) \\ &= [P(x \leq x)]^n \\ &= [F(x)]^n \end{aligned}$$

2.1 Generalised Extreme value (GEV) distribution

We can attempt to standardise the values of X_M by finding a sequence of constants $\alpha_1, \alpha_2, \ldots$ and $\beta_1, \beta_2, \ldots > 0$ so that the limiting distribution:

$$\lim_{n\to\infty} P\left(\frac{x_M - \alpha_n}{\beta_n} \leq x\right) = \lim_{n\to\infty} [F(\beta_n x + \alpha_n)]^n$$

depends only on x.

For example, if the individual losses are distributed exponentially with $F(x) = 1 - e^{-\lambda x}$, we can set $\alpha_n = \frac{1}{\lambda} \ln n$ and $\beta_n = \frac{1}{\lambda}$.

Let's see how we can simplify this:

2.1 Generalised Extreme value (GEV) distribution

Let $X \sim \mathrm{Exp}(\lambda), \alpha_n = \frac{1}{\lambda} \ln n$ and $\beta_n = \frac{1}{\lambda}$ for all n.
By substituting in for α_n and β_n and by using the CDF of the exponential distribution, we have

$$\lim_{n\to\infty} P\left(\frac{x_M - \alpha_n}{\beta_n} \le x\right) = \lim_{n\to\infty} [F(\beta_n x + \alpha_n)]^n$$

(Hint: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$.)

$$\begin{aligned}
\lim_{n\to\infty} [F(\beta_n x + \alpha_n)]^n &= \lim_{n\to\infty} \left[F\left(\frac{1}{\lambda}x + \frac{1}{\lambda}\ln n\right)\right]^n \\
&= \lim_{n\to\infty} \left\{1 - \exp\left[-\lambda\left(\frac{1}{\lambda}x + \frac{1}{\lambda}\ln n\right)\right]\right\}^n \\
&= \lim_{n\to\infty} \{1 - \exp(-x - \ln n)\}^n \\
&= \lim_{n\to\infty} \left\{1 - \frac{e^{-x}}{n}\right\}^n = e^{-e^{-x}}
\end{aligned}$$

2.1 Generalised Extreme value (GEV) distribution

The last line in the above simplification follows from the hint:

$$\lim_{n\to\infty}\left\{1-\frac{e^{-x}}{n}\right\}^n = \lim_{n\to\infty}\left\{1+\left(\frac{-e^{-x}}{n}\right)\right\}^n = e^{-e^{-x}}$$

This distribution is known as the standard Gumbel distribution.

The standard Gumbel distribution is a particular type of extreme value distribution.

2.1 Generalised Extreme value (GEV) distribution

GEV distribution

More generally, whatever the underlying distribution of the data, the distribution of the standardised maximum values will converge to a distribution called the generalised extreme value (GEV) distribution as n increases, ie $\lim_{n\to\infty}[F(\beta_n x+\alpha_n)]^n = H(x)$.

The cumulative distribution function of the GEV distribution is:

$$H(x) = \begin{cases} \exp\left(-\left(1+\dfrac{\gamma(x-\alpha)}{\beta}\right)^{-\frac{1}{\gamma}}\right) & \gamma \neq 0 \\ \exp\left(-\exp\left(-\dfrac{(x-\alpha)}{\beta}\right)\right) & \gamma = 0 \end{cases}$$

2.1 Generalised Extreme value (GEV) distribution

This distribution has three parameters:

- a location parameter α
- a scale parameter $\beta > 0$
- a shape parameter γ.

The parameters α and β just rescale (shift and stretch) the distribution. They are analogous to (but do not usually correspond to) the mean and standard deviation.

The parameter γ determines the overall shape of the distribution (analogous to the skewness) and its sign (positive, negative or zero) results in three different shaped distributions.

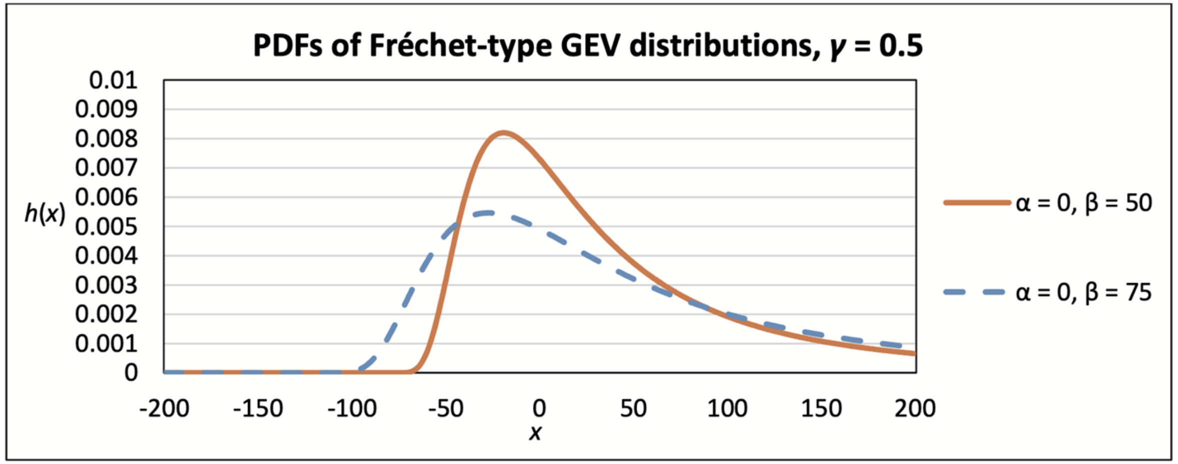
2.1 Generalised Extreme value (GEV) distribution

1. Fréchet-type GEV distribution

For $\gamma > 0$, the distribution is a Fréchet-type GEV distribution. Earlier, we derived the PDF as:

$$h(x) = \frac{1}{\beta}\left(1 + \frac{\gamma(x-\alpha)}{\beta}\right)^{-\left(1+\frac{1}{\gamma}\right)} \exp\left(-\left(1 + \frac{\gamma(x-\alpha)}{\beta}\right)^{-\frac{1}{\gamma}}\right)$$

Fréchet-type GEV distributions



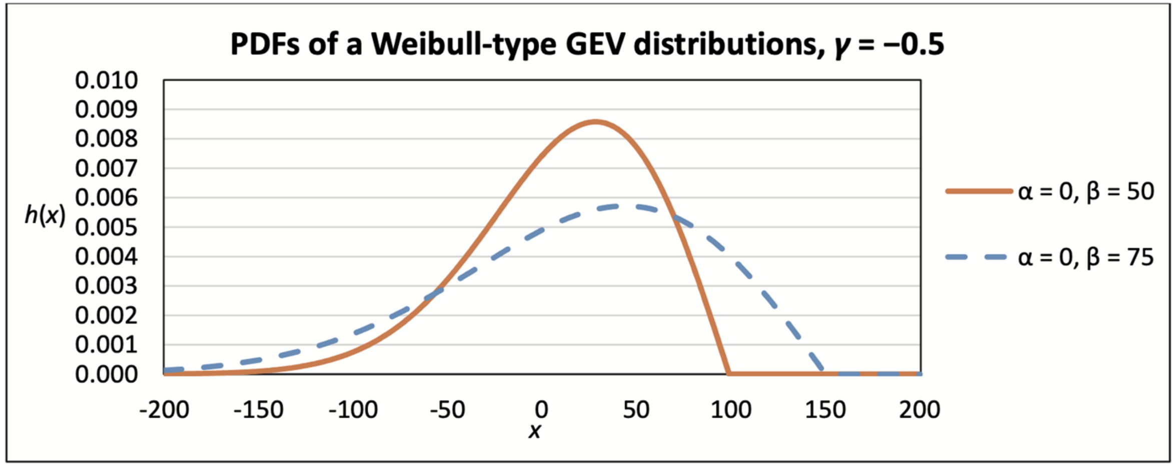
2.1 Generalised Extreme value (GEV) distribution

2. Weibull-type GEV distribution

For $\gamma < 0$, the distribution is a Weibull-type GEV distribution. The PDF is of the same form as the Fréchet-type GEV distribution, ie it is given by:

$$h(x) = \frac{1}{\beta}\left(1 + \frac{\gamma(x-\alpha)}{\beta}\right)^{-\left(1+\frac{1}{\gamma}\right)} \exp\left(-\left(1 + \frac{\gamma(x-\alpha)}{\beta}\right)^{-\frac{1}{\gamma}}\right)$$

Weibull-type GEV distribution



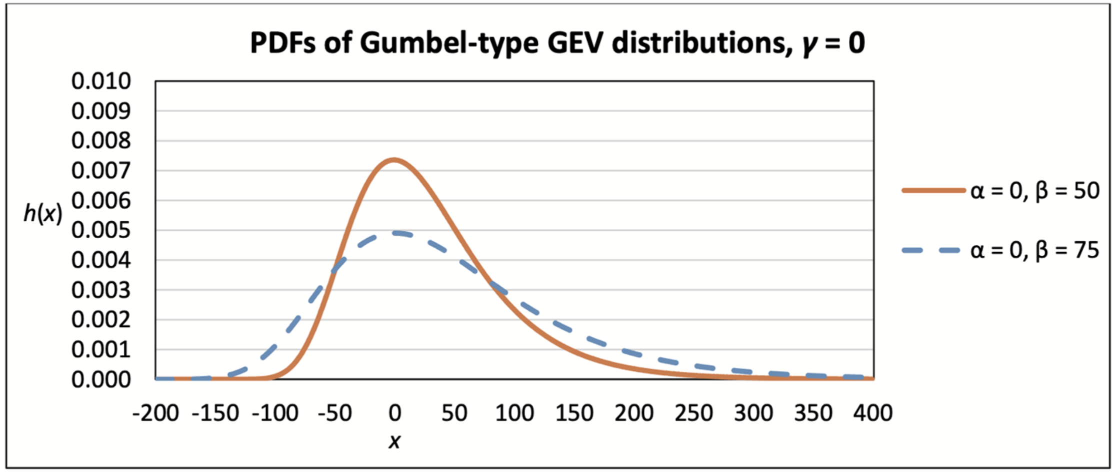
2.1 Generalised Extreme value (GEV) distribution

3. Gumbel-type GEV distribution

When $\gamma = 0$, the GEV distribution reduces to the Gumbel distribution. In this case, the PDF is given by:

$$h(x) = \frac{1}{\beta}\exp\left(-\left[\frac{(x-\alpha)}{\beta} + \exp\left(-\frac{(x-\alpha)}{\beta}\right)\right]\right)$$

Gumbel-type GEV distributions



2.1 Choosing the form of the GEV distribution

	GEV distributions (for the maximum value) corresponding to common loss distributions		
Type	**WEIBULL**	**GUMBEL**	**FRÉCHET**
Shape parameter	$\gamma < 0$	$\gamma = 0$	$\gamma > 0$
Underlying distribution	Beta Uniform Triangular	Chi-square Exponential Gamma Lognormal Normal Weibull	Burr F Log-gamma Pareto t
Range of permitted values	$x < \alpha - \frac{\beta}{\gamma}$	$-\infty < \boldsymbol{x} < \infty$	$x > \alpha - \frac{\beta}{\gamma}$

Question

CS2A S2021 Q1

An Analyst is assessing the risks of an equity portfolio and wishes to estimate the probability that the portfolio will incur at least one daily loss exceeding 5% next month.

Explain how a generalised extreme value distribution and the block maxima method could be used to estimate this probability.

Solution

Collect daily returns and group into months
Take the maximum loss each month and remove all other data
Find the parameters for the GEV distribution
using maximum likelihood estimation
Calculate 1 - H(0.05), where H(x) is the cumulative distribution function of the GEV distribution
which gives the probability that the maximum daily loss that month will exceed 5%

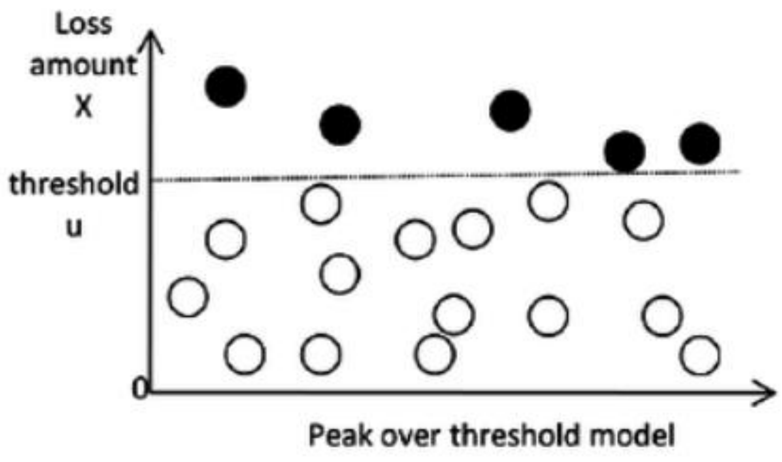
3 Peak Over Threshold Exceedances

Generalized Pareto Distribution

As an alternative to focusing on the maximum value, we can **consider the distribution of all the values of the variable that exceed some (large) specified threshold,** eg all claims exceeding £1 million.

For large samples, whatever the underlying distribution, the distribution of the threshold exceedances will converge to the **generalised Pareto distribution.**

This enables us to model the tail of a distribution by selecting a suitably high threshold and then fitting a generalised Pareto distribution to the observed values in excess of that threshold.



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Peak Over Threshold Exceedances

Generalized Pareto Distribution

Threshold exceedances

X – u/ X > u ***where u is the threshold***

The higher the value of u, the more extreme values we have of X. However, using a higher threshold means that we have fewer values with which to fit the extreme value distribution.

3 Peak Over Threshold Exceedances

Generalized Pareto Distribution - CDF

If the maximum possible value of X is $x_F \leq \infty$, the cumulative distribution function of the excess is (for $0 \leq x < x_F - u$):

$$F_u(x) = P(X - u \leq x \mid X > u) = \frac{P(X - u \leq x, X > u)}{P(X > u)}$$

$$= \frac{P(X \leq x + u, X > u)}{P(X > u)}$$

$$= \frac{P(X \leq x + u) - P(X \leq u)}{P(X > u)}$$

$$= \frac{F(x + u) - F(u)}{1 - F(u)}$$

3 Peak Over Threshold Exceedances

Generalized Pareto Distribution - CDF

More generally we find that, whatever the underlying distribution of the data, the distribution of the threshold exceedances will converge to a generalised Pareto distribution as the threshold u increases, ie $\lim_{u\to\infty} F_u(x) = G(x)$.

The generalised Pareto distribution is a two-parameter distribution with CDF:

$$G(x) = \begin{cases} 1 - \left(1 + \dfrac{x}{\gamma\beta}\right)^{-\gamma} & \gamma \neq 0 \\ 1 - \exp\left(-\dfrac{x}{\beta}\right) & \gamma = 0 \end{cases}$$

This distribution has two parameters:

- a scale parameter $\beta > 0$
- a shape parameter γ.

When $\gamma = 0$, this distribution reduces to the exponential distribution.

4 Measures of Tail Weights

There are a number of measures we can use to quantify the tail weight of a particular distribution, ie how likely very large values are to occur.

Tail weight is a measure of how quickly the (upper) tail of a PDF tends to 0.

We will consider four ways of measuring tail weight:

1. the existence of moments
2. limiting density ratios
3. the hazard rate
4. the mean residual life.

4 Measures of Tail Weights

Existence of moments

Recall that the k th moment of a continuous positive-valued distribution with density function $f(x)$ is:

$$\int_0^\infty x^k f(x)dx$$

If more number of non central moments exist, then that distribution has a lighter tail.

4 Measures of Tail Weights

Limiting density ratios

We can compare the thickness of the tail of two distributions by calculating the relative values of their density functions at the far end of the upper tail.

We calculate this as:

$$\lim_{x \to \infty} \frac{f_{x1}(x)}{f_{x2}(x)}$$

- **If the ratio equals ∞, then the numerator has a heavier tail.**
- **If the ratio equals 0, then the denominator has a heavier tail.**

4 Measures of Tail Weights

Limiting density ratios

For example, if we compare the Pareto distributions with parameters $\alpha = 2$ and $\alpha = 3$ (both with the same value of λ), we find that:

$$\lim_{x\to\infty} \frac{f_{\alpha=2}(x)}{f_{\alpha=3}(x)} = \lim_{x\to\infty} \left\{ \frac{2\lambda^2}{(\lambda+x)^3} \Big/ \frac{3\lambda^3}{(\lambda+x)^4} \right\} = \frac{2}{3\lambda} \lim_{x\to\infty} (\lambda + x) = \infty$$

This confirms that the distribution with $\alpha = 2$ has a much thicker tail.

If we compare the gamma distribution with the Pareto distribution, we find that the presence of the exponential factor in the gamma density results in a limiting density ratio of zero, which confirms that the gamma distribution has a lighter tail.

4 Measures of Tail Weights

Hazard rate

The hazard rate of a distribution with density function $f(x)$ and distribution function $F(x)$ is defined as:

$$h(x) = \frac{f(x)}{1 - F(x)}$$

We can interpret the hazard rate by analogy with μ_x , the force of mortality at age x .

- **If the force of mortality increases as a person's age increases, relatively few people will live to old age (corresponding to a light tail).**
- **If, on the other hand, the force of mortality decreases as the person's age increases, there is the potential to live to a very old age (corresponding to a heavier tail).**

4 Measures of Tail Weights

Mean residual life

The mean residual life of a distribution with density function $f(x)$ and distribution function $F(x)$ is defined as:

$$e(x) = \frac{\int_x^\infty (y-x)f(y)dy}{\int_x^\infty f(y)dy} = \frac{\int_x^\infty \{1-F(y)\}dy}{1-F(x)}$$

This function gives the expected remaining survival time given survival up until this point.

- **If MRT is an increasing function of x, then it corresponds to fatter tail.**
- **If MRT is a decreasing function of x, then it corresponds to lighter tail.**

Question

CS2A A2023 Q6

A hydroelectric company is managing a water reservoir created from a dam in a river valley. The dam was originally chosen so that the water level would exceed a threshold of 50 metres in about 2 days in every 300 days. In these extreme events, the excess water is left to escape the reservoir so that the water level is kept below the safety 50-metre limit.

It is believed that the daily water level in the reservoir follows an exponential distribution with mean μ.

(i) Estimate the value of μ. [3]

(ii) Determine the expected threshold exceedance of the water level over the 50-metre threshold. [2]

Contd..

Question

CS2A A2023 Q6

In order to better manage the excess water, it is now assumed that the excess water level follows a Generalised Pareto distribution with scale parameter $\beta = 1$.

(iii) Explain the circumstances in which the Generalised Pareto distribution would be preferred to the exponential distribution. [2]

(iv) Estimate the value of the parameter γ if the expected threshold exceedance is the same as that in part (ii). [3]
[Total 10]

Solution

(i)
Since the exponential distribution with parameter \lambda and with expectation \mu=1/\lambda has tail probability
Exp(-x/ \mu) then [1]
Exp(-50/\mu)=2/300 so
-50/\mu =log(2/300)=-5.010635 [1]
So
\mu=-50/5.010635=9.978775 [1]

(ii)
Since the threshold exceedance distribution for the exponential distribution is the same as the original distribution then [1]
[or since the exponential distribution is memoryless, then …]
the random variable U=X-50|X>50 has the same expectation as above, i.e. 9.978775 [1]

Solution

(iii)
GPD is preferred if extreme weather events are becoming more likely [1]
and therefore the exceedance distributions are expected to have fatter tails than those of the exponential [1]
modelling of the tails is seen as more important in a scenario such as this [1]
other sensible comments contrasting the GPD and the exponential [1]
[Total marks 4, maximum 2]

(iv)
If beta =1 the Pareto distribution will have expectation the same as the expected exceedance amount
\gamma /(gamma -1)= 9.978775 [1]
or
\gamma = (gamma -1)*9.978775 [1]
\gamma= 9.978775/(9.978775-1)= 1.111374 [1]
[Total 10]