IAQS INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: SRM - 1

Chapter: Unit 3 & 4

Category: Assignment Solutions

1.

i) The chi-squared test is a suitable overall test.

Test statistic = $\frac{\sum(Observed\ number\ of\ death - Expected\ number\ of\ deaths)^2}{Expected\ number\ of\ deaths}$

Age group	Central exposed to risk	Number of Deaths in the sample	Standard mortality rate used	Expected number of deaths	Observed - Expected	[(Observed - Expected) ^ 2] / Expected
20-24	56655	80	0.000937	53.09	26.91	13.64
25-29	61220	78	0.000934	57.18	20.82	7.58
30-34	64908	80	0.001042	67.63	12.37	2.26
35-39	62052	85	0.001358	84.27	0.73	0.01
40-44	58751	120	0.001969	115.68	4.32	0.16
45-49	54900	150	0.003168	173.92	-23.92	3.29
50-54	48679	295	0.00555	270.17	24.83	2.28
55-59	41699	366	0.008925	372.16	-6.16	0.10
						29.32

The test statistic ~ Chi square with m degrees of freedom
Here m is the number of age groups, which in this case is 8. We have not calculated any parameters. Hence, m remains 8

The critical value of the chi-squared distribution at the 5% level of significance with 8 degrees of freedom is 15.51

Value of test statistic = 29.32
Given, Test statistic > Critical value, we reject the null hypothesis

ii) Thus, standard mortality rates are not a good representation of the actual mortality experience and it is recommended that the company performs experience analysis to set its mortality assumptions.

2. Three methods of graduation that the life company can use, along with their respective advantages and disadvantages are described below:

Graduation by fitting Parametric Formula: we assume that mortality can be modelled using a mathematical formula

Advantages

• Suitable with reasonably large experience data to be able to fit a parametric formula to the crude rates: The insurance company has been selling term assurances for a number of years and is considered 'large' so it is likely that the company would have sufficiently large data to consider this approach
• The rates will automatically be smooth
• It is easy to identify the mortality trends if the same formula is used
• The goodness of fit is usually satisfactory
• Calculations can be computerised
• Can give most weight to the ages where most data was available.

Disadvantages

• It is often difficult to find a single formula that fits over the whole age range
• If the formula used does not include enough parameters, it will not be flexible enough to follow the crude rates closely, which may result in over-graduation. If too many parameters are included, sudden bends may appear in the graduated curve, which may result in undergraduation.
• It can be very time consuming, even with computerisation.
• For practical use, it may not be sufficient to choose and fit a formula using statistical methods alone. It will also be necessary to inspect the results in the light of previous knowledge of mortality experience, especially at very young and very old ages where the data may be scarce. Therefore, it may be necessary to adjust the graduation to obtain a satisfactory final result.

Graduation by reference to a Standard Table: we assume that there is a simple relationship between the observed mortality and an appropriate standard table

Advantages

• This will be particularly useful if we do not have much data from experience in which we are interested
• the method can give good results on very scanty data
• you usually do not have to bother testing for smoothness
• knowledge of other tables is automatically brought into the graduation
• there should be little difficulty with the ends of the table, ie amount of extrapolation required is limited

Disadvantages

• reliability of results can be doubtful if there is little data (although this is true for other graduation methods as well)
• it is not always possible to find a suitable standard table (and thus adherence to data would be poor if this were the case)
• any errors in the original table will be repeated.

Graduation using spline functions - (To be explained from notes)

3.

a) If the graduated rates are smooth but show little adherence to the data, then we say that the data may be over-graduated.

Similarly, if the graduated rates follow the crude rates closely but result in an irregular progression of over ages, we say that the data is under-graduated.

Therefore, to test the graduated rates for over/under-graduation, we need to test for both smoothness and adherence to data.

Chi-square test

We can test adherence to data using the chi square test.

The null hypothesis is:

H0: the graduated rates are the true underlying mortality rates for the population.

We can calculate the individual standardized deviations at each age using the approximation:

$$z_x = \frac{\theta_x - E_x\dot{q}_x}{\sqrt{E_x\dot{q}_x(1-\dot{q}_x)}} \approx \frac{\theta_x - E_x\dot{q}_x}{\sqrt{E_x\dot{q}_x}}$$

The approximation holds because $(1-\dot{q}_x) \approx 1$ for all x since the $\dot{q}_x$ terms are small.

Thus,

Age, x	E_x	θ_x	$\dot{q}_x$	z_x	$z_x^{\ 2}$
20-24	120	1	0.1515%	2.65	7.03
25-29	5982	12	0.2089%	-0.14	0.02
30-34	27839	65	0.2731%	-1.26	1.60
35-39	35487	124	0.3442%	0.17	0.03
40-44	40859	156	0.4223%	-1.26	1.59
45-49	39850	220	0.5075%	1.25	1.56
50-54	34859	189	0.6000%	-1.39	1.94
55-59	29349	210	0.7000%	0.32	0.10
				0.33	**13.87**

The test statistic for the chi-squared test is:

$$\sum z_x^2 = 13.87$$

Since the graduation was carried out graphically, we lose 2 or 3 degrees of freedom for every 10 age groups included in the graduation. We were given data from 8 age groups, so we are left with about 6 degrees of freedom.

From the tables, we find that the upper 5% point of χ_6^2 is 12.59.

As the value of test statistic exceeds this, we reject the null hypothesis and conclude that the graduated rates do not provide a good fit to the data.

Test for smoothness

To test for smoothness, we can calculate the third differences of the graduated quantities.

The third differences can be calculated as follows:

Age, x	$\dot{q}_x$	$\Delta\dot{q}_x = \dot{q}_{x+1} - \dot{q}_x$	$\Delta^2\dot{q}_x = \Delta\dot{q}_{x+1} - \Delta\dot{q}_x$	$\Delta^3\dot{q}_x = \Delta^2\dot{q}_{x+1} - \Delta^2\dot{q}_x$
20-24	0.1515%	0.0574%	0.0068%	0.0001%
25-29	0.2089%	0.0642%	0.0069%	0.0001%
30-34	0.2731%	0.0711%	0.0070%	0.0001%
35-39	0.3442%	0.0781%	0.0071%	0.0002%
40-44	0.4223%	0.0852%	0.0073%	0.0002%
45-49	0.5075%	0.0925%	0.0075%	
50-54	0.6000%	0.1000%		
55-59	0.7000%			

The criterion of smoothness usually used is that the third differences of the graduated rates should:
a. be small in magnitude compared with the quantities themselves; and
b. progress regularly.

For this graduation, both these conditions are met, which indicates that the graduated rates are very smooth.

Conclusion From the above two tests, we can see that the graduated rates: -
do not meet the chi square test for adherence to data; and -
are smooth.

Thus, the graduation seems to have led to the data being over-graduated.

b) We concluded above that the rates were over-graduated based on the observation that the chi-square test indicated that the graduated rates do not adhere to the data. However, a closer look at the standardized deviations indicates that more than half the value of test statistic came from the first age group of 20-24 years [7.03 out of a test statistic of 13.87]. The data in respect of this age group is particularly scanty (only one death in the last ten years!); therefore the crude rates could well be unreliable. Ignoring this outlier, the graduated rates in fact show a very good adherence to data, as the remaining standardized deviations are quite small. In this case, the graduation seems to be neither over nor undergraduated but may be considered adequate; as the graduation is both smooth as well as adheres well to the crude data (except for the dodgy first age group data).

Using the chi-square test to determine the adherence to data in case of graphical graduation has a limitation because it is necessary to determine the number of degrees of freedom for the test statistic. The number of degrees of freedom to use when graphical graduation has been used is not obvious.

The graduating curve has to a certain extent been forced to fit the rough data but it is subjective as to how many degrees of freedom should be deducted for this.

The chi square test is thus approximate and any result should be considered intelligently and not just blindly accepted. [19]

4.
i) Methods of graduation are:
a) Fitting a parametric formula: use the third differences of the graduated rates which should be small in magnitude and progress regularly to ensure smoothness of rates
b) Graduation using spline functions
c) Standard table: Standard table is already acceptably smooth.

ii) Reasons for smoothing
· By smoothing the experience, we can make use of data at adjacent ages to improve the estimates at each age.
· We believe that mortality varies smoothly with age. Therefore the crude estimate of mortality at any age carries information about mortality at adjacent ages.
· To remove sampling (or random) errors.
· To produce a smooth set of rates that are suitable for a particular purpose such as premium for life insurance contracts.
· Any irregularities, jumps and anomalies in financial quantities (such as premiums for life insurance contracts) are hard to justify to customers. (2)

iii) Appropriate method of graduation
a) The male members covered under PMJJY (Pradhan Mantri Jeevan Jyoti Yojna) With reference to a standard table, because there are many extant tables dealing with male population covered between ages 18 & 60 years.

b) The female population of a large developing country. By parametric formula, because the experience is large (or because the graduated rates may form a new standard table for the country).

c) The third gender patients suffering from brain cancer in a particular region. Graphical, because no suitable standard table is likely to exist and the experience is small. (3)

iv) To test for overall goodness of fit we use the χ2 test.

The null hypothesis is that the graduated rates are the same as the true underlying rates for the block of business.

The test statistic

$$\chi_n^2 = \sum_{x}^{all\ ages} Z_x^2$$

where n is the degrees of freedom.

Age	Exposed to risk	Observed Death	Graduated Rates (qx)	Expected Death	Z_x	Z_x^2
50	1280	4	0.00230	2.9440	0.6155	0.3788
51	2030	5	0.00262	5.3186	-0.1381	0.0191
52	1950	11	0.00296	5.7720	2.1761	4.7353
53	2160	7	0.00331	7.1496	-0.0559	0.0031
54	2480	10	0.0037	9.1760	0.2720	0.0740
55	1455	7	0.00415	6.0383	0.3914	0.1532
56	2100	11	0.00463	9.7230	0.4095	0.1677
57	1865	17	0.00518	9.6607	2.3613	5.5757
58	1990	16	0.00578	11.5022	1.3262	1.7588
59	1725	9	0.00645	11.1263	-0.6374	0.4063

Total 6.7204 13.2720

Observed Test Statistic is 13.27

The number of age groups (n) is 10, but we lose an unknown number of degrees for the graduation, perhaps 2. So $n = 8$, say.

The critical value of the chi-squared distribution with 8 degrees of freedom at the 5% level is 15.51.

Since 13.27 < 15.51

We do not reject the null hypothesis. **(4)**

5. (i) “Undergraduation” occurs when too much emphasis is given to goodness of fit. Undergraduated rates adhere closely to the crude rates, but the resulting rates do not show a smooth progression from age to age.

“Overgraduation” occurs when too much emphasis is given to smoothness. Overgraduated rates show a smooth progression from age to age, but the resulting rates do not adhere closely to the crude rates.

(ii)

The chi-squared test is for the overall fit of the graduated rates to the data.

The test statistics is $\sum z_x^2$, where

$$z_x = \frac{(\theta_x - E_x \overset{o}{q}_x)}{\sqrt{E_x \overset{o}{q}_x (1 - \overset{o}{q}_x)}}$$

However, considering the fact that q_x is very small, we redefined the z_x as below

$$z_x \approx \frac{(\theta_x - E_x \overset{o}{q}_x)}{\sqrt{E_x \overset{o}{q}_x}}$$

The calculations are given in the table below:

Average Age	Θx	qx	qx	Expected Deaths	Zx	zx^2
23	2	0.00889	0.00220	1.98	0.01421	0.00020
28	4	0.00833	0.00240	2.88	0.65997	0.43556
33	5	0.00923	0.00260	3.38	0.88116	0.77645
38	7	0.01000	0.00360	5.40	0.68853	0.47407
43	8	0.01091	0.00540	5.94	0.84523	0.71441
48	9	0.01500	0.00900	7.20	0.67082	0.45000
53	9	0.01385	0.01500	9.75	-0.24019	0.05769
58	5	0.01429	0.02300	8.05	-1.07498	1.15559

$$\sum z_x^2 = 4.06397$$

The test statistic has a chi-squared distribution with degrees of freedom given by number of age groups less 1 for the parametric function and further reduction for using the standard table.

The critical value of chi-squared distribution with 6 degree freedom at 5% level is 12.59.

Since 4.06397 < 12.59, there is no evidence to reject the null hypothesis that the graduated rates are the true rates underlying the crude rates.

(iii)

Signs test

a. The Signs test looks for overall bias

b. If the null hypothesis is true, the number of positive signs is distributed Binomial (8, 0.5). From the table, we observe that there are 6 positive signs.

Prob (observed number of positive signs <=6) = 1- Prob(positive signs >6)

$= 1 - \left\{\binom{8}{7} + \binom{8}{8}\right\} * (0.5)^8$ = 1- 0.035156 = 0.965

This is greater than 0.025 (two tailed test)

c. We cannot reject the null hypothesis and conclude that the graduated rates are not systematically higher or lower than the crude rates.

Grouping of Signs test

a. The grouping of signs test looks for run or clumping of deviations with the same sign for overgraduation.

b. We have total 8 age groups with 6 positive signs and 2 negative signs. There is only one run in this analysis.

Pr (one positive run) =

$$\frac{\left\{\binom{5}{0}\binom{3}{1}\right\}}{\binom{8}{6}}$$ =3/28=0.10714

This is greater than 0.05 (using one-tailed test).

c. We accept the null hypothesis that graduated rates are true underlying the crude rates.

[Total 14]

6. Lee Carter Model

$$\log_e m_{x,t} = a_x + b_x k_t + \varepsilon_{x,t}$$

where:

$m_{x,t}$ is the central mortality rate at age x in year t,

a_x describes the general shape of mortality at age x,

b_x measures the change in the rates in response to an underlying time trend in the level of mortality k,

k_t reflects the effect of the time trend on mortality at time t, and

$\varepsilon_{x,t}$ are iid random variables with means of zero and constant variance.

To obtain unique parameter estimates for this model, important constraints are imposed.

The usual constraints are that $\sum_x b_x = 1$ and $\sum_t k_t = 0$.

7. i)

- produce a smooth set of rates that are suitable for a particular purpose
- remove random sampling errors
- use the information available from adjacent ages.
- make them fit for the purpose for which they are intended

ii)

Age x	Exposed to Risk	Observed deaths	Graduated mortality rates	Expected deaths	z_x	z_x^2
35	5444	80	0.01658	90	-1.0801	1.1666
36	5355	102	0.01787	96	0.6446	0.4156
37	5268	88	0.01894	100	-1.1789	1.3898
38	5197	110	0.01988	103	0.6575	0.4324
39	4978	91	0.02022	101	-0.9624	0.9262
40	4831	106	0.02154	104	0.1902	0.0362
41	4654	123	0.02365	110	1.2327	1.5196
42	4521	107	0.02811	127	-1.7817	3.1744
43	4487	122	0.02957	133	-0.9272	0.8598
44	4321	125	0.03069	133	-0.6610	0.4369
45	4101	140	0.03081	126	1.2142	1.4742
46	4021	145	0.03166	127	1.5683	2.4596
47	3951	140	p			Sum: 14.2912

$X^2_{(13-4)}$ test-Statistic = 21.67 (Upper-tail @ 1%)
$Z_x = (O-E)/\text{sqrt}(E)$

$\sum Z_x^2 = 14.2912 + (140-3951p)^2 / 3951p$

X^2 test-statistic degree of freedom: 13 – 4
Critical-value is 21.67
$\therefore$ $14.2912 + (140-3951p)^2 / 3951p < 21.67$
$\rightarrow$ $(140-3951p)^2 / 3951p < 29.153.64p$
$\rightarrow$ $15610401p^2 – 1135433.64p + 19600$
Range of p € (0.028, 0.044)

iii) - Cumulative Deviations test
- Signs test
- Grouping of Signs test
- Serial Correlations test

iv) - One factor in the age-period-cohort is linearly dependent on the other two
- Intense data demands, for example, for all age groups, more than 100 years of data may be required

8. i) Graduation refers to the process of using statistical techniques to improve the estimates provided by the crude rates. The aims of graduation are
· To produce a smooth set of rates that are suitable for a particular purpose
· To remove random sampling errors
· To use the information available from adjacent ages [2]

ii) The three desirable features of graduation are
(a) smoothness
(b) adherence to data and
(c) suitability for the purpose in hand. [1]

iii) If the graduation process results in rates that are smooth but show little adherence to the data, then we say that the data may be overgraduated. Here the graduated rates may not be representative of the underlying experience.

Undergraduation refers to the case where insufficient smoothing has been carried out. This will tend to produce a curve of inadequate smoothness, but better adherence to data. In this case, the graduated rates will follow the crude rates very closely, but will show an irregular progression over the range of ages. [2]

iv) The graduated rates are derived using the formulae given and the estimated parameters, as given below:

Age	Male	Female
20	0.000574	0.000337
25	0.000849	0.000499
28	0.001074	0.000632
33	0.001587	0.000935
66	0.020633	0.012362
71	0.030221	0.018202

v) The appropriateness of graduation can be tested using chi-square test. The exposed to risk at each age can be taken as 1,000 lives. Based on this exposed-to-risk, the actual and expected deaths can be estimated.
The difference between the observed deaths and expected deaths as per the graduated rates are summarised below.

As the actual deaths for females at age 20 and 25 are below 5, these 2 cells are combined.

	Age	Actual	Expected	(A-E)	(A-E)^2	(A-E)^2/E
Males	20	5.69	0.5742	5.1158	26.1714	45.5792
	25	7.94	0.8491	7.0909	50.2805	59.2142
	28	11.06	1.0737	9.9863	99.7256	92.8777
	33	15.41	1.5875	13.8225	191.0625	120.3570
	66	21.51	20.6325	0.8775	0.7700	0.0373
	71	29.98	30.2212	-0.2412	0.0582	0.0019
Females	20-25	8.05	0.8358	7.2142	52.0441	62.2658
	28	6.51	0.6315	5.8785	34.5567	54.7212
	33	9.04	0.9351	8.1049	65.6887	70.2443
	66	12.55	12.3615	0.1885	0.0355	0.0029
	71	17.44	18.2025	-0.7625	0.5813	0.0319

The observed $\chi 2$ statistic is higher than $\chi 2$ with 11 degrees of freedom. Hence we can conclude that the graduation process is not appropriate.

[3]

9. Expectations have been used in mortality forecasting in the form of expert opinion: an assumed forecast or scenario is specified, often accompanied by alternative high and low scenarios. Most official statistical agencies have based estimates on this approach in the past.

An approach under expectations is projection of mortality using reduction factors.

$R_{x,t}$ - which measures the proportion by which the mortality rate at age x, qx, is expected to be reduced by future year t.

We have a reduction factor equation as:

$$R_{x,t} = \alpha_x + (1-\alpha_x)(1-f_{n,x})^{t/n}$$

where and α_x and $f_{n,x}$ represent, respectively, the ultimate reduction factor and the proportion of the total decline $(1-\alpha_x)$ assumed to occur in n years. The approach embodies an exponential decline in mortality over time to the asymptotic value α_x, and uses expert opinion to set the targets α_x and $f_{n,x}$.

The future mortality is then projected using the below formula and some chosen base year.

$$q_{x,t} = q_{x,0} \times R_{x,t}$$

10. i) The graph is an example of overgraduation [1]

ii) To test smoothness, we need to calculate the third differences of the graduated quantities. The third differences measure the change in curvature. The criterion of smoothness usually used is that the third differences of the graduated quantities should: a. be small in magnitude compared with the quantities themselves; and b. progress regularly. [3]

iii) The x2 -test will fail to detect several defects that could be of considerable financial importance.

a. There could be a few large deviations offset by a lot of very small deviations.

b. The graduation might be biased above or below the data by a small amount. The x2 statistic can often fail to detect consistent bias if it is small.

c. Even if the graduation is not biased as a whole, there could be significant groups of consecutive ages (called runs or clumps) over which it is biased up or down. This is still to be avoided. [3]

iv) Signs test [1]