INSTITUTE OF ACTUARIAL & QUANTITATIVE STUDIES

Subject: SRM - 2

Chapter: Unit 1 & 2

Category: Assignment Solutions

1.

(i) For the Pareto distribution with parameters α, λ as per the tables we have:

$$E(X) = \frac{\lambda}{\alpha - 1}$$

And

$$Var(X) = \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = E(X)^2 \frac{\alpha}{\alpha-2}$$

And so

$$E\left(X^2\right) = Var(X) + E(X)^2 = E(X)^2\left(\frac{\alpha}{\alpha-2}+1\right) = E(X)^2\left(\frac{2\alpha-2}{\alpha-2}\right)$$

The observed values we are trying to fit are

$$E(X) = 170$$
$$E\left(X^2\right) = 400^2 + 170^2 = 434.626^2$$

So we have

$$\frac{2\alpha-2}{\alpha-2} = \frac{E(X^2)}{E(X)^2} = \frac{434.626^2}{170^2} = 6.53633$$

And so

$$\alpha = \frac{2 - 2\times 6.53633}{(2-6.53633)} = 2.441$$

And finally $\lambda = 1.441 \times 170 = 244.95$

(ii) We must solve

$$0.5 = 1 - \left(\frac{244.95}{244.95 + x}\right)^{2.441}$$

Re-arranging and taking roots gives

$$0.5^{\frac{1}{2.441}} = 0.7527965 = \frac{244.95}{244.95 + x}$$

And so

$$x = \frac{244.95 - 244.95 \times 0.7527965}{0.7527965} = 80.44$$

So the median is significantly lower than the mean. This demonstrates how skew the Pareto distribution is.

2.

(i) (a) Let the parameters of the Lognormal distribution be μ and σ.

Then we must solve

$$e^{\mu+\frac{\sigma^2}{2}} = 230 \qquad \text{(A)}$$

$$e^{2\mu+\sigma^2}(e^{\sigma^2}-1) = 110^2 \qquad \text{(B)}$$

$$\text{(B)} \div \text{(A)}^2 \Rightarrow e^{\sigma^2}-1 = \frac{110^2}{230^2}$$

so $\quad e^{\sigma^2} = 1+\dfrac{110^2}{230^2} = 1.22873$

so $\quad \sigma^2 = \log 1.22873 = 0.205984$

so $\sigma = 0.45385$

Substituting into (A) gives

$$e^{\mu+\frac{0.205984}{2}} = 230$$

$$\mu = \log(230) - \frac{0.205984}{2}$$

$$= 5.3351$$

(b) This time we have

$$e^{\mu+0.6745\sigma} = 510 \qquad (A)$$

$$e^{\mu-0.6745\sigma} = 80 \qquad (B)$$

$\log A + \log B \Rightarrow 2\mu = \log 510 + \log 80$

so $\mu = 5.30822$

and substituting into (A)

$$5.30822 + 0.6745\sigma = \log 510$$

$$\sigma = \frac{\log 510 - 5.30822}{0.6745} = 1.37315$$

(ii) Calculating the upper and lower quartiles from the parameter in (i)(a) gives

$$UQ = e^{5.3351+0.6745\times0.45385} = 282 \qquad \text{cf } 510$$

$$LQ = e^{5.3351-0.6745\times0.45385} = 153 \qquad \text{cf } 80$$

This is not a good fit, suggesting the underlying claims have greater weight in the tails than a Lognormal distribution.

3.

(i) Let the parameters be c and γ as per the tables.

Then we have:

$$1-e^{-c\times 20^{\gamma}}=0.1 \text{ so } \quad e^{-c\times 20^{\gamma}}=0.9 \quad \text{and so } c\times 20^{\gamma}=-\log 0.9 \text{ (A)}$$

And similarly $c\times 95^{\gamma}=-\log 0.1$ (B)

(A) divided by (B) gives $\left(\frac{20}{95}\right)^{\gamma}=\frac{\log 0.9}{\log 0.1}=0.0457575$

So $\gamma=\frac{\log 0.0457575}{\log\left(\frac{20}{95}\right)}=1.9795337$

And substituting into (A) we have $c=-\frac{\log 0.9}{20^{1.9795337}}=0.000280056$

(ii) The 99.5th percentile loss is given by

$$1-e^{-0.00280056x^{1.9795337}}=0.995$$

So that $-0.000280056x^{1.9795337}=\log 0.005$

$$\log x=\frac{\log\left(\frac{\log 0.005}{-0.000280056}\right)}{1.9795337}=4.97486366$$

So $x=e^{4.97486366}=144.73$

4.

(i) $M_S(t) = E(e^{tS})$

$= E(E(e^{t(X_1+X_2+\cdots+X_N)} | N)$

$= E(M_X(t)^N)$ since the X_i are independent and identically distributed

$= E(e^{N \log M_X(t)})$

$= M_N(\log M_X(t))$

$= \exp(\lambda(\exp(\log(M_X(t)-1))))$

$= \exp(\lambda(M_X(t)-1))$

(ii) $M'_S(t) = M_S(t) \times \lambda M'_X(t)$

$E(S) = M'_S(0) = M_S(0) \times \lambda \times M'_X(0)$

$= 1 \times \lambda \times \mu$

$= \lambda\mu$

$M''_S(t) = M'_S(t) \times \lambda \times M'_X(t) + M_S(t) \times \lambda \times M''_X(t)$

$E(S^2) = M''_S(0) = M'_S(0) \times \lambda \times M'_X(0) + M_S(0) \times \lambda \times M''_X(0)$

$= \lambda\mu \times \lambda \times \mu + 1 \times \lambda \times (\sigma^2 + \mu^2)$

$= \lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2$

And so

$$\mathrm{Var}(S) = E(S^2) - E(S)^2$$
$$= \lambda^2\mu^2 + \lambda\mu^2 + \lambda\sigma^2 - \lambda^2\mu^2$$
$$= \lambda(\mu^2 + \sigma^2)$$

(iii) First, we must calculate the mean and variance of a single claim, say Y. Let us denote by X the underlying loss. Then

$$
\begin{aligned}
E(Y) &= \int_0^{200} 0.01xe^{-0.01x}dx + \int_{200}^{\infty} (x+50)\times 0.01\times e^{-0.01x}dx \\
&= \int_0^{\infty} 0.01xe^{-0.01x}dx + 50\int_{200}^{\infty} 0.01e^{-0.01x}dx \\
&= E(X) + 50\times P(X>200) \\
&= 100 + 50\times e^{-200\times 0.01} \\
&= 100 + 6.76676 \\
&= 106.76676
\end{aligned}
$$

$$E(Y^2) = \int_0^{200} 0.01x^2e^{-0.01x}dx + \int_{200}^{\infty} (x+50)^2\times 0.01e^{-0.01x}dx$$

$$= \int_0^{\infty} 0.01x^2e^{-0.01x}dx + \int_{200}^{\infty} xe^{-0.01x}dx + 50^2\int_{200}^{\infty} 0.01e^{-0.01x}dx$$

$$= E(X^2) + \left[-100xe^{-0.01x}\right]_{200}^{\infty} + \int_{200}^{\infty} 100e^{-0.01x}d + 50^2P(X>200)$$

$$= 100^2 + 100^2 + 20,000e^{-2} + \left[-100^2e^{-0.01x}\right]_{200}^{\infty} + 2,500e^{-2}$$

$$= 20,000 + 20,000e^{-2} + 10,000e^{-2} + 2,500e^{-2}$$

$$= 20,000 + 32,500e^{-2}$$

$$= 24,398.39671$$

And finally, using the results from part (ii)

$$E(S) = 500E(X) = 500 \times 106.76676$$
$$= 53,383.38$$

and

$$\text{Var}(S) = 500E(X^2) = 500 \times 24,398.39671 = 12,199,198.36$$

5.

(i) (a) $E(N) = E[E(N|\mu)]$
$= E[\mu] = 2/8 = 0.25$

(b) $\text{var}(N) = E[\text{var}(N|\mu)] + \text{var}[E(N|\mu)]$
$= E[\mu] + \text{var}[\mu]$
$= 2/8 + 2/8^2 = 0.28125$

(ii) Let Y be aggregate claims from one policy.

Individual claim is gamma with $\alpha = 2$ and $\lambda = 0.001$.

$$E(Y) = E(X)E(N) = 2000 \times 0.25 = 500.$$

$$Var(Y) = E(N)Var(X) + Var(N)E(X)^2$$

$$= 0.25 \times 2000000 + \tfrac{9}{32} \times 2000^2 = 1,625,000.$$

So the mean and variance of total claims are 500,000 and 1,625,000,000 respectively.

(iii) Our approximate distribution for S is $S \sim N(500{,}000\ ,\ 1625000000)$.

$$P(S > 550000) = P\left(Z > \frac{550000 - 500000}{\sqrt{1625000000}}\right) = P(Z > 1.24035) = 0.1074\,.$$

(iv) The prob three years in a row is $0.1074^3 = 0.00124$.

The probability of this happening is very low. It is more likely that the insurance company's belief about the distribution of claims amounts is incorrect.

The normal approximation tails off quickly and so underestimates the probability of extreme events

6. (i)

(a) To protect itself from the risk of large claims.

(b)

- Excess of loss reinsurance where the reinsurer pays any amount of a claim above the retention.
- Proportional reinsurance where the reinsurer pays a fixed proportion of any claim.

(ii) We must first find the parameters α and λ of the Pareto distribution.

$$\frac{\lambda}{\alpha-1} = 270 \text{ and } \frac{\alpha\lambda^2}{(\alpha-1)^2(\alpha-2)} = 340^2$$

$$\frac{\alpha}{\alpha-2} \times \frac{\lambda^2}{(\alpha-1)^2} = 340^2$$

$$\text{so } \frac{\alpha}{\alpha-2} = \frac{340^2}{270^2} = 1.585733882$$

$$\text{so } \alpha = \frac{2 \times 1.585733882}{1.585733882 - 1}$$

$$= 5.4145$$

and $\lambda = 270 \times 4.4145 = 1191.920375$

We need to find M such that $P(X > M) = 0.05$

i.e. $\left(\frac{\lambda}{\lambda + M}\right)^{\alpha} = 0.05$

$$\frac{\lambda}{\lambda + M} = 0.05^{\frac{1}{\alpha}}$$

$$\lambda = 0.05^{\frac{1}{\alpha}}(\lambda + M)$$

$$M = \frac{\lambda\left(1 - 0.05^{\frac{1}{\alpha}}\right)}{0.05^{\frac{1}{\alpha}}}$$

$$= \frac{1191.920375 \times \left(1 - 0.05^{\frac{1}{5.4145}}\right)}{0.05^{\frac{1}{5.4145}}}$$

$$= \underline{\underline{880.8}}$$

7.

(i) Let the loss amount be X. Then

$E(X) = 0.8 \times 100 + 0.2 \times 115 = 103$

$E(X^2) = 0.8 \times (100^2 + 400) + 0.2\,(115^2 + 900) = 11{,}145$

$\text{Var}(X) = E(X^2) - E(X)^2 = 11{,}145 - 103^2 = 536$

(ii) No, the loss distribution is not Normal. To see this, note that (for example) the pdf of the combined distribution will have local maxima at both 100 and 115. [Consider the case where the variances are very small to see this]

(iii) $\Pr(X > 130) = 0.8 \times \Pr(N(100, 20^2) > 130) + 0.2 \times \Pr(N(115,30^2) > 130)$

$$= 0.8 \times \Pr\left(N(0,1) > \frac{130-100}{20}\right) + 0.2 \times \Pr\left(N(0,1) > \frac{130-115}{30}\right) \Pr(X >$$
$$130) = 0.8 \times \Pr(N(100, 20^2) > 130) + 0.2 \times \Pr(N(115, 30^2) > 130)$$

$$= 0.8 \times \Pr(N(0,1) > 1.5) + 0.2 \times \Pr(N(0,1) > 0.5)$$
$$= 0.8 \times (1 - 0.93319) + 0.2 \times (1 - 0.69146)$$

$$= 0.115156$$

(iv) The relevant proportion is given by:

$$\frac{0.2 \times (1 - 0.69146)}{0.115156} = 53.6\%$$

8.

(i) Let X represent the distribution of individual claims. Let π denote the probability that an individual claim involves the reinsurer. Then

$$\pi = P(X > M) = \int_M^\infty \lambda e^{-\lambda x} dx$$

$$= \left[-e^{-\lambda x} \right]_M^\infty$$

$$= e^{-\lambda M}$$

(ii) $M_{I_i}(t) = E(e^{tI_i}) = \pi e^t + 1 - \pi = e^{t-\lambda M} + 1 - e^{-\lambda M}$

(iii) Using the results for the moment generating function of a compound distribution, we have

$$M_K(t) = M_N(\log M_{I_i}(t))$$

$$= \left(pM_{I_i}(t) + 1 - p \right)^n$$

$$= \left(p(\pi e^t + 1 - \pi) + 1 - p \right)^n$$

$$= \left(p\pi e^t + p - p\pi + 1 - p \right)^n$$

$$= \left(p\pi e^t + 1 - p\pi\right)^n$$

$$= \left(pe^{t-\lambda t} + 1 - pe^{-\lambda t}\right)^n$$

Which is the MGF of a binomial distribution with parameters n and $p\pi$.

Hence, by the uniqueness of MGFs K has a binomial distribution with parameters n and $p\pi$.

9.

(i) Let the individual loss amounts have distribution X. Then

$$E(X) = \int_0^{100} 0.01333xe^{-0.01333x}dx + 100 \times P(X > 100)$$

$$= \left[-xe^{-0.01333x}\right]_0^{100} + \int_0^{100} e^{-0.01333x}dx + 100\int_{100}^{\infty} 0.01333e^{-0.01333x}dx$$

$$= -100e^{-1.333} + \left[-75e^{-0.01333x}\right]_0^{100} + 100\left[-e^{-0.01333x}\right]_{100}^{\infty}$$

$$= -100e^{-1.333} - 75e^{-1.333} + 75 + 100e^{-1.333}$$

$$= 55.2302$$

Hence $E(S) = 50 \times 55.2302 = 2761.5$

$$E(X^2) = \int_0^{100} 0.01333x^2 e^{-0.01333x} dx + 100^2 P(X > 100)$$

$$= \left[-x^2 e^{-0.01333x} \right]_0^{100} + \int_0^{100} 2xe^{-0.01333x} dx + 100^2 e^{-1.333}$$

$$= -100^2 e^{-1.333} + \left[-\frac{2x}{0.01333} e^{-0.01333x} \right]_0^{100} + \int_0^{100} \frac{2}{0.01333} e^{-0.01333x} dx + 100^2 e^{-1.333}$$

$$= -\frac{200}{0.01333} e^{-1.333} + \left[-\frac{2}{0.01333^2} e^{-0.01333x} \right]_0^{100}$$

$$= -\frac{200}{0.01333} e^{-1.333} - \frac{2}{0.01333^2} e^{-1.333} + \frac{2}{0.01333^2}$$

$$= 4330.6$$

and so

$$Var(S) = 50 \times 4330.6 = 216529 = (465.33)^2$$

(ii) (a) The normal distribution is $N(2761.5, 465.33^2)$

(b) The Log-Normal distribution has parameters μ and σ with

$$E(S) = e^{\mu+\sigma^2/2}$$
$$Var(S) = e^{2\mu+\sigma^2}(e^{\sigma^2}-1) = E(S)^2 \times (e^{\sigma^2}-1)$$

So substituting gives

$$216529 = 2761.5^2 \times (e^{\sigma^2}-1)$$

$$e^{\sigma^2} = \frac{216529}{2761.5^2} + 1 = 1.028394$$

$$\sigma^2 = \log(1.028394) = 0.027998$$

$$\sigma = 0.167327$$

And now we can substitute for σ to give

$$2761.5 = e^{\mu+0.027998/2}$$

$$\mu = \log(2761.5) - 0.027998/2 = 7.90953$$

(iii) Using the Normal distribution:

$$P(N(2761.5, 465.33^2) > 3000) = P\left(N(0,1) > \frac{3000-2761.5}{465.33}\right)$$

$$= P(N(0,1) > 0.51) = 1-0.69497 = 0.30503$$

From tables.

Using the log-normal distribution,

$$P(\log N(7.90953, 0.167327^2) > 3000) = P(N(7.90953, 0.167327^2) > \log(3000))$$

$$= P(N(0,1) > \frac{\log 3000 - 7.90953}{0.167327}).$$

$$= P(N(0,1) > 0.58) = 1-0.71904 = 0.28096$$

10.

Let the annual number of claims be denoted by N. Then

$$P(N=k) = \int_0^\infty P(N=k|\mu)\, f(\mu)\, d\mu$$

$$= \int_0^\infty e^{-\mu} \frac{\mu^k}{k!} \lambda e^{-\lambda\mu} d\mu$$

$$= \frac{\lambda}{k!} \int_0^\infty \mu^k e^{-(1+\lambda)\mu} d\mu$$

$$= \frac{\lambda}{k!} \times \frac{\Gamma(k+1)}{(1+\lambda)^{k+1}} \int_0^\infty \frac{(1+\lambda)^{k+1}}{\Gamma(k+1)} \mu^k e^{-(1+\lambda)\mu} d\mu$$

$$= \frac{\lambda}{(1+\lambda)^{k+1}} \times 1$$

Where the final integral is 1 since the integrand is the pdf of a Gamma distribution.

So

$$P(N=k) = \frac{\lambda}{(1+\lambda)^{k+1}} = \frac{\lambda}{1+\lambda} \times \frac{1}{(1+\lambda)^k}, \text{ for k} = 0,\ 1,\ 2,\ \ldots$$

Which means that N has a geometric distribution with parameter $p = \dfrac{\lambda}{1+\lambda}$. This is equivalent to a Type II negative binomial with k=1